On Some Equations of the Duffing Type

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der
Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

von

Cyrine Fitouri

aus

Tunesien

Promotionskomitee

Prof. Dr. Michel Chipot (Vorsitz)
Prof. Dr. Stefan Sauter
Prof. Dr. G. Vergara-Caffarelli (Begutachter)

Zürich, 2008

Die vorliegende Arbeit wurde von der Mathematisch-naturwissenschaftlichen Fakultät der Universität Zürich auf Antrag von Prof. Dr. M. Chipot und Prof. Dr. S. Sauter als Dissertation angenommen.

Zusammenfassung

Wir untersuchen in dieser Arbeit eine bestimmte Klasse von Differentialgleichungen zweiter Ordnung, die sogenannten Duffing Gleichungen. Mit Hilfe dieser Gleichungen wird zum Beispiel das chaotische Verhalten mechanischer Systeme in doppelten Potentialfelder beschrieben. Zuerst untersuchen wir hinreichende Bedingungen unter welchen eine Gleichung diesen Typs beschränkte Lösungen besitzt. Dadurch versuchen wir eigentlich das chaotische Verhalten solcher beschränkter Lösungen zu unterbinden. Weiter erhalten wir im grossen Dumping-Fall nahezu optimale Schranken für die Lösung der Gleichungen vom Duffing Typ. Schliesslich finden wir scharfe Abschätzungen für beschränkte Lösungen bestimmter semilinearer dissipativer Gleichungen zweiter Ordnung.

Abstract

In this thesis, we study some type of differential equations of second order, called Duffing equations. These equations are used to describe the chaotic behavior of a mechanical system in a double potential field. First, we study sufficient conditions under which an equation of this type has bounded solutions. Basically, we try to prevent the chaotic behavior of such bounded solutions. Secondly, we obtain a close-to-optimal bound of the solution of equations of the Duffing type, in the large dumping case. Finally, we find the sharp estimates of bounded solutions of certain semilinear second order dissipative equations.

Acknowledgements

Many individuals have influenced the work described in this thesis. I am very grateful to my advisor Prof. Dr. Michel Chipot for his continued help and support and to whom I am very much obliged. I am also very grateful to Prof. Dr. Alain Haraux from whom I learned a lot. For being an external referee, and for his interest in this work, I would like to thank Prof. G. Vergara-Caffarelli. I express my thanks to all the secretaries for their help. I wish also to thank my collaborator Sana, my colleagues (Anna, Abigail, Karen, Sorin, Pia...) and all my friends (Warda, Hanen...). Special thanks for Sanda, Frau Kirta Suss, Herr Walter Suss, Karin, Mira and Abada who helped me a lot since I am in Zurich. I want to thank Nadia. Thank you mama, papa, Yassine and Marwen.

To Papa ♡

To Mama ♡

To Yassine.

Contents

Introduction

The Duffing equation was first introduced by the electrical engineer G. Duffing in 1918. It is a nonlinear differential equation which describes the chaotic behavior of a mechanical system in a double potential field.

In this thesis we are interested essentially in the bounded solutions of some equations of the Duffing type.

The first chapter is devoted to the proof of the existence of exactly 3 different bounded solutions which approche any solution of the equation $u'' + cu' + |u|^p u - u = f(t)$ as t tends to infinity, c large enough and under a smallness condition on the bounded forcing term f. For small values of c, we establish the existence of exactly 3 different T- periodic solutions when f is T-periodic and satisfies a smallness condition.

In the second chapter we use differential inequalities to improve in the strongly damped case the estimate of W.S. Loud for the ultimate bound of the solutions to the second order ODE

$$u'' + cu' + g(u) = f(t)$$

where $c > 0$, $f \in L^\infty([t_0, +\infty))$ and $g \in C^1(\mathbb{R})$ satisfies some conditions.

Using the technique of the second chapter we improve in the third one, the $L^\infty(\mathbb{R}, V)$ bound of the unique bounded solution, in the strongly damped case of $u''+cu'+Au = f(t)$ whenever $A = A^* \geq \lambda I$ is a bounded or unbounded linear operator on a real Hilbert space H, $V = D(A^{1/2})$ and λ, c are positive constants, while $f \in L^\infty(\mathbb{R}, H)$.

In the last chapter we consider H, V two real Hilbert spaces such that $V \subset H$ with continuous and dense imbedding, and a convex function $F \in C^1(V)$. We obtain a close-to-optimal ultimate bound of the energy for solutions in $C^1(\mathbb{R}^+, V) \cap W^{2,\infty}_{loc}(\mathbb{R}^+, V')$, to $u'' + cu' + bu + \nabla F(u) = f(t)$ whenever $f \in L^\infty(\mathbb{R}, H)$.

Chapter 1

On a general equation of Duffing's type with double well potential

1 Introduction.

Under a smallness condition on f and for c large enough, A.Haraux [11] proved that we can have for the system

$$u'' + cu' + u^3 - u = f(t),$$

where $c > 0$ and $f \in L^\infty(\mathbb{R})$, a behavior less chaotic then the one described in [5, 8, 16, 18]. In this chapter we determine a smallness condition on f which depends on $p \geq 2$ and which gives the same results for the more general equation

$$u'' + cu' + |u|^p u - u = f(t). \tag{1.1}$$

The methods of the proofs are the same as in [11] with more technical difficulties especially for global results (Theorem 1.5 and 1.6).
We improve in particular slightly the result of Theorem 1.5 of [11].

2 The linear case.

Consider the equation

$$u'' + cu' + \lambda u = f = \mathcal{L}u, \tag{1.2}$$

with $f \in L^\infty(\mathbb{R})$, $c > 0$ and $\lambda \neq 0$.
Let r_i, $i = 1, 2$ be the roots of $r^2 + cr + \lambda = 0$. Then when $r_1 \neq r_2$ a particular solution of (1.2) is given by

$$v(t) = \frac{1}{r_1 - r_2}\{e^{r_1 t}\int_0^t e^{-r_1 s} f(s)ds - e^{r_2 t}\int_0^t e^{-r_2 s} f(s)ds\} \ \forall t \in \mathbb{R}.$$

indeed setting

$$w_i(t) = e^{r_i t} \int_0^t e^{-r_i s} f(s) ds,$$

one has

$$\begin{aligned} {w_i}'(t) &= r_i e^{r_i t} \int_0^t e^{-r_i s} f(s) ds + f(t), \\ (w_1 - w_2)'(t) &= r_1 e^{r_1 t} \int_0^t e^{-r_1 s} f(s) ds - r_2 e^{r_2 t} \int_0^t e^{-r_2 s} f(s) ds, \\ (w_1 - w_2)''(t) &= {r_1}^2 e^{r_1 t} \int_0^t e^{-r_1 s} f(s) ds - {r_2}^2 e^{r_2 t} \int_0^t e^{-r_2 s} f(s) ds + (r_1 - r_2) f(t). \end{aligned}$$

Thus we get

$$(w_1 - w_2)'' + c(w_1 - w_2)' + b(w_1 - w_2) = (r_1 - r_2) f,$$

which shows that v is a particular solution.
Then the general solution of (1.2) is given by

$$W(t) = \frac{1}{r_1 - r_2}\{A - \int_0^t e^{-r_1 s} f(s) ds\} e^{r_1 t} - \frac{1}{r_1 - r_2}\{B - \int_0^t e^{-r_2 s} f(s) ds\} e^{r_2 t}, \tag{1.3}$$

where A and B are contants.
Now, our aim is to show that if we assume w bounded then A and B are fixed, which means that the equation (1.2) has a unique bounded solution.

•<u>Suppose $f \geq 0$</u>.

The roots r_1 and r_2 are given by

$$r_1 = -\frac{c}{2} + \sqrt{\frac{c^2}{4} - \lambda} \quad \text{and} \quad r_2 = -\frac{c}{2} - \sqrt{\frac{c^2}{4} - \lambda}.$$

1) Suppose $0 < \lambda < \frac{c^2}{4}$.
Then the two roots are negative and W is bounded near $+\infty$ for every A and B. This follows from

$$0 \leq e^{r_i t} \int_0^t e^{-r_i s} f(s) ds \leq e^{r_i t} (-\frac{1}{r_i}) e^{r_i s}|_0^t \|f\|_\infty = -\frac{1}{r_i}\{1 - e^{r_i t}\}\|f\|_\infty.$$

To be bounded near $-\infty$ requires

$$A = \int_0^{-\infty} e^{-r_1 s} f(s) ds \ , \ B = \int_0^{-\infty} e^{-r_2 s} f(s) ds.$$

The solution is then given by

$$\mathcal{L}^{-1}(f) = \int_{-\infty}^{t} \frac{e^{-r_1(s-t)} - e^{-r_2(s-t)}}{r_1 - r_2} f(s)ds. \tag{1.4}$$

Clearly this is bounded near $-\infty$ thanks to the estimate

$$0 \leq e^{r_i t} \int_{-\infty}^{t} e^{-r_i s} f(s)ds \leq e^{r_i t}(-\frac{1}{r_i})e^{-r_i s}|_{-\infty}^{t} \|f\|_\infty = -\frac{1}{r_i}\|f\|_\infty.$$

2) Suppose $\lambda < 0$.
Then $r_1 > 0$ and $r_2 < 0$. Arguing as above we see that W is bounded if and only if

$$A = \int_0^{+\infty} e^{-r_1 s} f(s)ds \qquad B = \int_0^{-\infty} e^{-r_2 s} f(s)ds,$$

which gives us a unique solution

$$\mathcal{L}^{-1}(f) = \frac{1}{r_1 - r_2}\{\int_t^{+\infty} e^{-r_1(t-s)} f(s)ds - \int_{-\infty}^{t} e^{-r_2(t-s)} f(s)ds\}. \tag{1.5}$$

3) Suppose $\lambda > \frac{c^2}{4}$.
Then we have

$$r_1 = -\frac{c^2}{4} + i\delta \quad \text{and} \quad r_2 = -\frac{c^2}{4} - i\delta,$$

where $\delta^2 = \lambda^2 - \frac{c^2}{4}$.
Arguing as above we have

$$\mathcal{L}^{-1}(f) = \int_{-\infty}^{t} \frac{e^{-r_1(s-t)} - e^{-r_2(s-t)}}{r_2 - r_1} f(s)ds. \tag{1.6}$$

In the case f is arbitary we see by decomposing f into f^+, f^- that (1.4) (1.5) and (1.6) give a bounded solution to (1.2). This is the only one since any other would be obtained by adding $Ce^{r_1 t} + De^{r_2 t}$ which is unbounded unless $C = D = 0$.

Computation of $|\mathcal{L}^{-1}|$.
- Suppose $0 < \lambda < \frac{c^2}{4}$.

Then r_1 and r_2 are negative and (1.4) holds

$$\begin{aligned}
\mathcal{L}^{-1}(f) &= \frac{1}{r_2 - r_1}\int_{-\infty}^{t}\{e^{r_1(t-s)} - e^{r_2(t-s)}\}f(s)ds,\\
|\mathcal{L}^{-1}(f)| &\le \frac{1}{|r_2 - r_1|}\int_{-\infty}^{t}\{|e^{r_1(t-s)} - e^{r_2(t-s)}|\}|f(s)|ds,\\
&\le \frac{1}{r_1 - r_2}\int_{-\infty}^{t}\{e^{r_1(t-s)} - e^{r_2(t-s)}\}|f(s)|ds \qquad (r_2 < r_1 < 0),\\
&= \frac{1}{r_1 - r_2}\{e^{r_1 t}(-\frac{1}{r_1})e^{-r_1 s}|_{-\infty}^{t} + e^{r_2 t}(\frac{1}{r_2})e^{-r_2 s}|_{-\infty}^{t}\}\|f\|_\infty,\\
&= \frac{1}{r_1 - r_2}\{\frac{1}{r_2} - \frac{1}{r_1}\}\|f\|_\infty = \frac{1}{r_2 r_1}\|f\|_\infty = \frac{1}{\lambda}\|f\|_\infty.
\end{aligned}$$

Then $|\mathcal{L}^{-1}| = \frac{1}{\lambda}$ since the equality holds for $f \equiv 1$.

• Suppose $\lambda < 0$.
$\mathcal{L}^{-1}(f)$ is given by (1.5) and we have since $r_1 > 0$ and $r_2 < 0$,

$$\begin{aligned}
|\mathcal{L}^{-1}(f)| &\le \frac{1}{r_1 - r_2}\{\int_{t}^{+\infty} e^{r_1(t-s)}ds + \int_{-\infty}^{t} e^{r_2(t-s)}ds\}\|f\|_\infty\\
&\le \frac{1}{r_1 - r_2}(\frac{1}{r_1} - \frac{1}{r_2}) = \frac{-1}{r_1 r_2} = -\frac{1}{\lambda}.
\end{aligned}$$

The equality $|\mathcal{L}^{-1}| = -\frac{1}{\lambda}$ follows.
• Suppose $\lambda > \frac{c^2}{4}$.
We use the formula (1.6) to get

$$\mathcal{L}^{-1}(f) = \frac{1}{r_2 - r_1}\int_{-\infty}^{t}\{e^{r_1(t-s)} - e^{r_2(t-s)}\}f(s)ds,$$

with $r_1 = -\frac{c}{2} + i\delta$ and $r_2 = -\frac{c}{2} - i\delta$.
Thus

$$\begin{aligned}
\mathcal{L}^{-1}(f) &= \frac{1}{r_2 - r_1}\int_{-\infty}^{t} e^{-\frac{c}{2}(t-s)}\{e^{i(t-s)}\delta - e^{-i(t-s)}\delta\}f(s)ds,\\
&= -\frac{1}{\delta}\int_{-\infty}^{t} e^{-\frac{c}{2}(t-s)}\sin[(t-s)\delta]f(s)ds\\
\Rightarrow \mathcal{L}^{-1}(f) &\le \frac{1}{\delta}(\int_{0}^{+\infty} e^{-\frac{c}{2}}|\sin(\delta u)|du)\|f\|_\infty.
\end{aligned}$$

This last integral is computed in [11] and this leads to

$$|\mathcal{L}^{-1}|_{=}\frac{1}{\lambda}\coth\{\frac{c\pi}{2\sqrt{4\lambda - c^2}}\}.$$

3 Main results.

We collect here all the results that we will prove later. We assume $p \geq 2$.

Theorem 1.1. *Under the condition*

$$\|f\|_\infty < \frac{p}{(p+1)^{\frac{p+1}{p}}}, \tag{1.7}$$

the equation (1.1) has a unique solution $\omega_0 \in W^{2,\infty}(\mathbb{R})$ such that

$$\|\omega_0\|_\infty < \frac{1}{(p+1)^{\frac{1}{p}}}. \tag{1.8}$$

Theorem 1.2. *If $c \geq 2\sqrt{p}$ and*

$$\|f\|_\infty < p((\frac{2p+1}{p+1})^{\frac{p+1}{p}} - 2), \tag{1.9}$$

then the equation (1.1) has a unique solution ω_+ and a unique solution $\omega_- \in W^{2,\infty}(\mathbb{R})$ such that

$$\|\omega_+ - 1\|_\infty < (\frac{2p+1}{p+1})^{\frac{1}{p}} - 1, \qquad \|\omega_- + 1\|_\infty < (\frac{2p+1}{p+1})^{\frac{1}{p}} - 1. \tag{1.10}$$

Theorem 1.3. *If $c \leq 2\sqrt{p}$, assuming the additional smallness condtion*

$$\|f\|_\infty \leq (\frac{c\sqrt{p}}{2} + p + 1)[(\frac{c\sqrt{p}}{2(p+1)} + 1)^{\frac{1}{p}} - 1] - (\frac{c\sqrt{p}}{2(p+1)} + 1)^{\frac{p+1}{p}} := \eta(c), \tag{1.11}$$

then (1.1) has a unique solution ω_+ and a unique solution $\omega_- \in W^{2,\infty}(\mathbb{R})$ such that

$$\|\omega_+ - 1\|_\infty < (1 + \frac{c\sqrt{p}}{2(p+1)})^{\frac{1}{p}} - 1, \quad \|\omega_- + 1\|_\infty < (1 + \frac{c\sqrt{p}}{2(p+1)})^{\frac{1}{p}} - 1. \tag{1.12}$$

Theorem 1.4. *If $c \leq \sqrt{\frac{p}{2}}$, assuming*

$$\|f\|_\infty \leq (\frac{c\sqrt{p}}{\sqrt{2}} + p + 1)(\frac{c\sqrt{p}}{\sqrt{2}(p+1)} + 1)^{\frac{1}{p}} - (\frac{c\sqrt{p}}{\sqrt{2}(p+1)})^{\frac{p+1}{p}} - p - \frac{c\sqrt{p}}{\sqrt{2}} := \eta_1(c), \tag{1.13}$$

then the equation (1.1) has a unique solution ω_+ and a unique solution $\omega_- \in W^{2,\infty}(\mathbb{R})$ such that

$$\|\omega_+ - 1\|_\infty < (1 + \frac{c\sqrt{p}}{\sqrt{2}(p+1)})^{\frac{1}{p}} - 1, \quad \|\omega_- + 1\|_\infty < (1 + \frac{c\sqrt{p}}{\sqrt{2}(p+1)})^{\frac{1}{p}} - 1. \tag{1.14}$$

Theorem 1.5. *Under the conditions*

$$c \geq 2\sqrt{p}, \quad f \in C_b(\mathbb{R}), \quad \|f\|_\infty < \inf\{\frac{1}{6p\sqrt{2}}, \frac{c}{\sqrt{1+c^2}} \frac{\sqrt{3p-4}}{8\sqrt{3p}}\} \tag{1.15}$$

any solution u *of (1.1) on some halfline* $J = (t_0, +\infty)$ *is asymptotic to one of the* 3 *solutions* $\omega_0, \omega_+, \omega_-$ *as* $t \to +\infty$.

Corollary 1.1. *Under the hypotheses of Theorem* 1.5*, if* f *is almost periodic, (1.1) has exactly* 3 *almost periodic solutions* $\omega_0, \omega_+, \omega_-$. *Moreover if* f *is* T*- periodic then so are* $\omega_0, \omega_+, \omega_-$.

Corollary 1.2. *Under the hypotheses of Theorem* 1.5*, if* f *is* T*- periodic, then (1.1) has no subharmonic periodic solution.*

The last main result is restricted to T-periodic solutions.

Theorem 1.6. *Let* f *be bounded and* T*- periodic. Under the condition*

$$\begin{aligned} &\|f\|_\infty (1 + \frac{\sqrt{T}}{c} \sqrt{K\|f\|_\infty^p + (\frac{p^4(p+1)}{(p^2-p-1)} + 1)T}) \\ &< (\frac{p}{p+1}) \inf\{(1 + \frac{c\sqrt{p}}{2(p+1)})^{\frac{1}{p}} - 1, (\frac{2p+1}{p+1})^{\frac{1}{p}} - 1\} \end{aligned} \tag{1.16}$$

with $K = 2^{(1+\frac{p}{2}(p-3)^+)} \dfrac{p^{\frac{p}{2}}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p-2}{2}}(p+1)T}{(p^2-p+3)^{\frac{p-2}{2}} c^p} + 2^{(p-3)^+} p^2(p^2-1) \dfrac{T^{p-1}}{c^p}$
$+ p^p(p+1)T$
equation (1.1) has at most 3 T*-periodic solutions.*

Corollary 1.3. *Let* f *be bounded and* T*- periodic satisfying both smallness condition of Theorem* 1.1 *and theorem* 1.6*. Then (1.1) has exactly three* T*- periodic solutions.*

Remark 1.1. *If* $p = 2$*, the condition* (1.15) *of Theorem* 1.5 *improves the condition* (1.9) *of [11] by a factor* $\frac{2\sqrt{2}}{\sqrt{3}}$.

4 Existence of 3 bounded solutions for f small.

The goal of this section is to give a proof of theorems 1.1, 1.2, 1.3 and 1.4.
First we establish the existence of the "small" solution. We introduce the operator $\mathcal{L}$ defined on

$$X := L^\infty(\mathbb{R})$$

by

$$D(\mathcal{L}) = W^{2,\infty} = \{u \in C^1(\mathbb{R}), u, u', u'' \in L^\infty(\mathbb{R})\}$$
$$\forall u \in D(\mathcal{L}), \quad \mathcal{L}u = u'' + cu' - u.$$

So that a bounded solution u of (1.1) is just a solution of

$$\mathcal{L}u = u'' + cu' - u = f - |u|^p u.$$

Since $\mathcal{L}$ corresponds to the linear case with $\lambda = -1$ we have

$$||\mathcal{L}^{-1}||_{L(X)} = 1.$$

We write the previous equation as

$$u = \mathcal{L}^{-1}(f - |u|^p u).$$

Now the mapping

$$\mathcal{T}(v) = \mathcal{L}^{-1}(f - |v|^p v),$$

leaves invariant the ball

$$B_r = \{v \in X, ||v||_X \leqslant r\},$$

if

$$||f||_X + r^{p+1} \leqslant r.$$

In order to have this inequality for some r it is enough to choose f such that

$$||f||_X \leq \sup_{r>0}(r - r^{p+1}).$$

To calculate this supremum let us consider for $r > 0$

$$h(r) = r - r^{p+1}.$$

We have

$$h'(r) = 0 \Leftrightarrow r = r_0 = (\frac{1}{p+1})^{\frac{1}{p}}$$

and $h''(r) = -p(p+1)r^{p-1} < 0$, then h is concave therefore the supremum is achieved for r_0.
Thus it is sufficient to take

$$\|f\|_X < h(r_0) = \frac{p}{(p+1)^{\frac{p+1}{p}}}$$

to have for some r

$$\mathcal{T}B_r \subset B_r.$$

Moreover for any u and $v \in \mathcal{B}_r$ with $r < r_0$, we have

$$\begin{aligned}
\|\mathcal{T}(u) - \mathcal{T}(v)\|_X &= \|\mathcal{L}^{-1}(f - |u|^p u) - \mathcal{L}^{-1}(f - |v|^p v)\|_X \\
&= \|\mathcal{L}^{-1}(|v|^p v - |u|^p u)\|_X \\
&\le \||v|^p v - |u|^p u\|_X = \|(p+1)\theta_t{}^p(u-v)\| \qquad \text{with} \quad \theta_t \in (u,v) \\
&\le (p+1)\|\theta_t\|_X^p \|u-v\|_X \\
&\le (p+1)r^p\|u-v\|_X \\
&\le \alpha\|u-v\|_X \quad \text{with} \quad \alpha < 1 \quad (\text{since } r < r_0).
\end{aligned}$$

Then $\mathcal{T}$ is a X-contraction on B_r. So there is a unique fixed point w_0 of $\mathcal{T}$ in B_r which is the solution of our problem. In addition we have $||w_0||_X < (\frac{1}{p+1})^{\frac{1}{p}}$.
For the two other solutions, due to the odd character of the non-linearity, by changing f to $-f$, we just need to study the existence of a second bounded solution close to 1. Setting $u = 1 + v$, we are reduced to consider the equation:

$$v'' + cv' + pv = f - \gamma(v) \quad \text{where} \quad \gamma(s) = |1+s|^p(1+s) - (p+1)s - 1,$$

that we rewrite in the form:

$$v = \mathcal{L}^{-1}(f - \gamma(v)),$$

where

$$\mathcal{L}u = u'' + cu' + pu.$$

<u>Case 1</u>: $c \ge 2\sqrt{p}$.

Let us set

$$\mathcal{T}(v) = \mathcal{L}^{-1}(f - \gamma(v))$$

and

$$B_r = \{v \in X, ||v||_X \leqslant r\}.$$

We have

$$\mathcal{T}(v) \in B_r \Leftrightarrow \|\mathcal{L}^{-1}(f - \gamma(v))\|_X \le r. \tag{1.17}$$

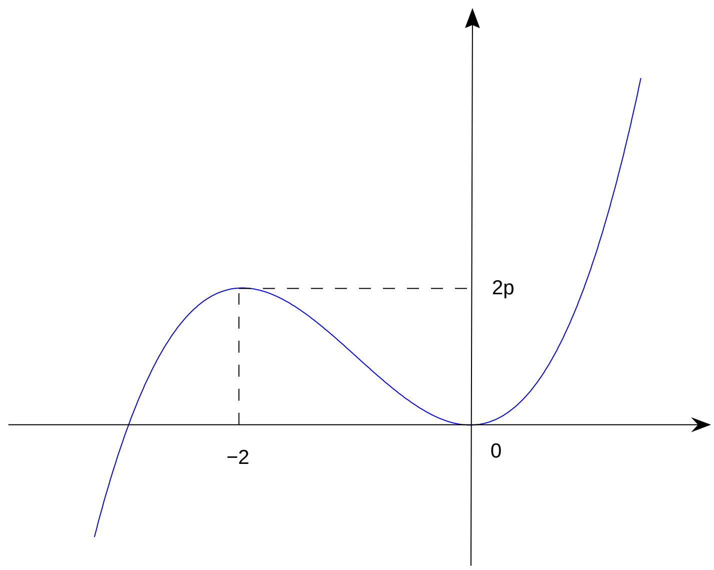
By the second section we have $||\mathcal{L}^{-1}|| = \frac{1}{p}$, and

$$\|\mathcal{L}^{-1}(f - \gamma(v))\|_X \le \frac{1}{p}\|f - \gamma(v)\|_X \le \frac{1}{p}\{\|f\|_X + \sup_{v \in B_r}\|\gamma(v)\|_X\}. \tag{1.18}$$

So to have (1.17) for any $v \in B_r$ it is enough that

$$\|f\|_X \le pr - \sup_{v \in B_r}\|\gamma(v)\|_X. \tag{1.19}$$

Now, our aim is to determine the supremum of $\|\gamma(v)\|_X$ when $v \in B_r$, for this at first let us study the behavior of $\gamma(s)$.
We have
$$\gamma(s) = |1+s|^p(1+s) - (p+1)s - 1$$
$$\gamma'(s) = (p+1)\{|1+s|^p - 1\}$$
hence the shape of the representative curve of γ is as follow:



Let us choose $r \leq 2$, then we have
$$\sup_{v\in B_r} \|\gamma(v)\|_X \leq \max(\gamma(r), \gamma(-r)) = \gamma(r). \tag{1.20}$$
Indeed to see that, set
$$h(r) = \gamma(r) - \gamma(-r) = (1+r)^{p+1} - (1-r)^{p+1} - 2(p+1)r \quad \forall p \geq 1$$
$$h'(r) = (p+1)[(1+r)^p + (1-r)^p - 2].$$
We have that for $p \geq 1$ the function $x \longrightarrow x^p$ is convex on $[0, +\infty[$ thus for any x and y in $[0, +\infty[$ we have
$$\left(\frac{x+y}{2}\right)^p \leq \frac{x^p}{2} + \frac{y^p}{2}.$$
Using this inequality we have
$$2 \leq (1+r)^p + (1-r)^p.$$

Then the function h is nondecreasing in addition $h(0) = 0$, this gives that $\gamma(r) \geq \gamma(-r)$ for $r \in [0, 2]$ and (1.20) follows. Thus (1.19) is equivalent for $r \leq 2$ to

$$\|f\|_X \leq pr - \gamma(r). \tag{1.21}$$

The maximum of $pr - \gamma(r)$ is achieved for $r_1 = (\frac{2p+1}{p+1})^{\frac{1}{p}} - 1$.

Let us show that $r_1 \leq 2$, set

$$l(p) = \ln(\frac{2p+1}{p+1}) - p\ln 2, \ \forall p \geq 1.$$

we have

$$l'(p) = \frac{1}{(2p+1)(p+1)} - \ln 2 \leq \frac{1}{6} - \ln 2 \leq 0, \quad \forall p \geq 1.$$

Thus the function l is decreasing and for $p = 1$

$$l(p) \leq l(1) = \ln\frac{3}{2} - \ln 2 < 0, \quad \forall p \geq 1.$$

So $l(p) \leq 0$ $\forall p \geq 1$. This is equivalent to

$$(\frac{2p+1}{p+1})^{\frac{1}{p}} \leq 2 \Leftrightarrow r_1 \leq 2.$$

So, let us choose now $r \in (0, r_1)$ and $\|f\|_X \leq pr - \gamma(r)$, then By (1.19) we have that $\mathcal{T}\mathcal{B}_r \subset \mathcal{B}_r$.
To prove that $\mathcal{T}$ is a contraction we first show that

$$|\gamma'(s)| < p \qquad \forall s \in [-r, r], \quad r < r_1. \tag{1.22}$$

Indeed one has

$$\gamma'(s) = (p+1)\{|1+s|^p - 1|\}.$$

Then for $0 \leq s < r_1$ it holds that

$$0 \leq \gamma'(s) < (p+1)\{|1+r_1|^p - 1\} = (p+1)\{\frac{2p+1}{p+1} - 1\} = p.$$

For $0 < s < 2$ one has

$$\begin{aligned}
\gamma'(-s) \leq 0 \quad \text{and} \quad \gamma'(-s) \geq -p &\Leftrightarrow (p+1)\{|1-s|^p - 1|\} \geq -p \\
&\Leftrightarrow |1-s|^p \geq \frac{1}{p+1} \\
&\Leftrightarrow 1 - s \geq (\frac{1}{p+1})^{\frac{1}{p}} \\
&\Leftrightarrow s \leq 1 - (\frac{1}{p+1})^{\frac{1}{p}} < 2.
\end{aligned}$$

This complete the proof of (1.22).
Then for any u and $v \in \mathcal{B}_r$ we have

$$\begin{aligned}
\|\mathcal{T}(u) - \mathcal{T}(v)\|_X &= \|\mathcal{L}^{-1}(f - \gamma(u)) - \mathcal{L}^{-1}(f - \gamma(v))\|_X \\
&= \|\mathcal{L}^{-1}(\gamma(u) - \gamma(v))\|_X \\
&\leq \frac{1}{p}\|\gamma(u) - \gamma(v)\|_X = \frac{1}{p}\|\gamma'(u + \theta_t(u-v))(u-v)\|_X \quad \text{for some } \theta_t \in]0,1[\\
&\leq \frac{1}{p}\sup_{[-r,r]}|\gamma'(s)|\|u - v\|_X \\
&\leq \alpha\|u - v\|_X \qquad \text{with} \qquad \alpha < 1
\end{aligned}$$

So, $\mathcal{T} : \mathcal{B}_r \to \mathcal{B}_r$ is a strict contraction. The fixed point of $\mathcal{T}$ is the positive bounded solution we looked for. To finish the proof in this case, two additional remarks are necessary.
1) First to justify the choice we made for $\|f\|_X$ in (1.9), we have

$$\begin{aligned}
pr_1 - \gamma(r_1) &= (2p+1)r_1 - (1+r_1)^{p+1} + 1 \\
&= (2p+1)[(\frac{2p+1}{p+1})^{\frac{1}{p}} - 1] - (\frac{2p+1}{p+1})^{\frac{p+1}{p}} + 1 \\
&= (\frac{2p+1}{p+1})^{\frac{p+1}{p}}(p+1) - (\frac{2p+1}{p+1})^{\frac{p+1}{p}} + 2p \\
&= p[(\frac{2p+1}{p+1})^{\frac{p+1}{p}} - 2] < p(\frac{1}{p+1})^{\frac{p+1}{p}}.
\end{aligned}$$

Indeed to see that let us write

$$\begin{aligned}
(\frac{2p+1}{p+1})^{\frac{p+1}{p}} - (\frac{1}{p+1})^{\frac{p+1}{p}} &= \frac{p+1}{p}\int_{\frac{1}{p+1}}^{\frac{2p+1}{p+1}} s^{\frac{1}{p}}ds \\
&= 2(\frac{p+1}{2p})\int_{\frac{1}{p+1}}^{\frac{2p+1}{p+1}} s^{\frac{1}{p}}ds \\
&< 2 \quad \text{(by Jensen's inequality since } s^{\frac{1}{p}} \text{ is concave for } p \geq 1).
\end{aligned}$$

2) The solution near 0 and the solution near 1 are distinct since the second one is greater than $1 - [(\frac{2p+1}{p+1})^{\frac{1}{p}} - 1] \geq (\frac{1}{p+1})^{\frac{1}{p}} > \|w_0\|_\infty$.

Indeed, for $p \geq 1$ the function $x \to x^p$ is convave.
Thus for any x_1, x_2, $x_1 \neq x_2$, we have

$$(\frac{x_1 + x_2}{2})^{\frac{1}{p}} \geq \frac{{x_1}^{\frac{1}{p}}}{2} + \frac{{x_2}^{\frac{1}{p}}}{2}$$

$$\Leftrightarrow \qquad {x_1}^{\frac{1}{p}} + {x_2}^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}(x_1 + x_2)^{\frac{1}{p}}.$$

Using this inequality we have

$$(\frac{2p+1}{p+1})^{\frac{1}{p}} + (\frac{1}{p+1})^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}} 2^{\frac{1}{p}} = 2.$$

Then the inequality follows.
Case 2 : If $c \leq 2\sqrt{p}$,

Then, by the second section and the Theorem 2.1 in [11], $||\mathcal{L}^{-1}|| \leq \frac{2}{c\sqrt{p}}$ and using the argument above we get

$$\mathcal{T}(v) = \mathcal{L}^{-1}(f - \gamma(v)),$$

leaves invariant the ball

$$B_r = \{v \in X, ||v||_X \leqslant r\},$$

if

$$\frac{2}{c\sqrt{p}}(||f||_X + \gamma(r)) \leq r.$$

This is satisfied for some postive r whenever

$$\|f\|_X \leq \sup_{r>0}(\frac{c\sqrt{p}}{2} r - \gamma(r)).$$

Set $\forall r > 0$

$$g(r) = \frac{c\sqrt{p}}{2} r - \gamma(r) = (p + \frac{c\sqrt{p}}{2} + 1)r - (1+r)^{p+1} + 1.$$

We have

$$g'(r) = 0 \Leftrightarrow r_2 = (\frac{c\sqrt{p}}{2(p+1)} + 1)^{\frac{1}{p}} - 1 \leq r_0,$$

in addition

$$g''(r) = -p(p+1)(1+r)^{p-1} < 0$$

then g is concave and it is maximum is achieved on r_2.
Thus it is enough to take

$$\|f\|_X \leq \sup_{r>0}(p + \frac{c\sqrt{p}}{2} + 1)r - (1+r)^{p+1} + 1 := M_1,$$

with

$$M_1 = (p + \frac{c\sqrt{p}}{2} + 1)r_2 - (1+r_2)^{p+1} + 1.$$

Moreover $\mathcal{T}$ is still a contraction on B_r for $r < r_2$ since

$$\begin{aligned}\|\mathcal{T}(u) - \mathcal{T}(v)\|_X &\leq \frac{2}{c\sqrt{p}}\|\gamma'(\theta_t)\|_X\{\|u-v\|_X\} \qquad \text{with } \theta_t \in (u,v)\\ &\leq \frac{2}{c\sqrt{p}}\sup_{[-r,r]}|\gamma'(s)|\{\|u-v\|_X\}.\end{aligned}$$

In an other hand we have for $s \in [-r, r]$

$$|\gamma'(s)| < \frac{c\sqrt{p}}{2}.$$

In fact,

$$\begin{aligned}\gamma'(s) < \frac{c\sqrt{p}}{2} &\Leftrightarrow (p+1)\{|1+s|^p - 1\} < \frac{c\sqrt{p}}{2}\\ &\Leftrightarrow |1+s| < (\frac{c\sqrt{p}}{2(p+1)} + 1)^{\frac{1}{p}} = r_2 + 1\\ &\Leftrightarrow s < r_2 \qquad (\text{since } s \geq -r > -r_2 > -r_0 \geq 1).\end{aligned}$$

Thus

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_X \leq \alpha\|u-v\|_X \quad \text{with} \quad \alpha < 1.$$

In addition the positive solution is still strictly greater than the small one 0, since

$$1 - [(\frac{c\sqrt{p}}{2(p+1)} + 1)^{\frac{1}{p}} - 1] > (\frac{1}{p+1})^{\frac{1}{p}}.$$

Then the final condition on f in this case is

$$\|f\|_X \leq (\frac{c\sqrt{p}}{2} + p + 1)[(\frac{c\sqrt{p}}{2(p+1)} + 1)^{\frac{1}{p}} - 1] - (\frac{c\sqrt{p}}{2(p+1)} + 1)^{\frac{p+1}{p}} := \eta(c).$$

Case 3 : If $c \leq \sqrt{\frac{p}{2}}$,
using the exact formula given by Theorem 2.1 in [11]:

$$\|\mathcal{L}^{-1}\|_{L(X)} = \frac{1}{p} \times \frac{1 + e^{\frac{-c\pi}{\sqrt{4p-c^2}}}}{1 - e^{\frac{-c\pi}{\sqrt{4p-c^2}}}} = \frac{1}{p}\coth\{\frac{-c\pi}{\sqrt{4p-c^2}}\},$$

we prove

$$\|\mathcal{L}^{-1}\|_{L(X)} \leq \frac{\sqrt{2}}{c\sqrt{p}}.$$

Indeed for p fixed, $c\|\mathcal{L}^{-1}\|_{L(X)}$ is an increasing function of c, hence for $c \leq \sqrt{\frac{p}{2}}$

$$c\|\mathcal{L}^{-1}\|_{L(X)} \leq \frac{1}{\sqrt{2p}} \times \frac{1 + e^{-\frac{\pi}{\sqrt{7}}}}{1 - e^{-\frac{\pi}{\sqrt{7}}}} < \sqrt{\frac{2}{p}},$$

since $\frac{\pi}{\sqrt{7}} > \ln 3$. The conclusion follows as in case 2 since $\frac{\sqrt{2}}{c\sqrt{p}} < \frac{2}{c\sqrt{p}}$.

5 Ultimate bound of the general solution.

In this section we derive a general, probably not optimal, estimate of the ultimate bound of general solution of (1.1).

Proposition 1.1. *For any solution u of* (1.1) *we have*

$$\overline{\lim_{t\to\infty}}\, |u|^{p+2}(t) \leq (\frac{p+2}{2})^{\frac{p+2}{2}} + (p+2)[\frac{c^2+(p+2)^2}{2p^2c^2}]\|f\|_\infty^2. \tag{1.23}$$

Proof. We introduce the energy $E(t) = \frac{1}{2}u'^2 + \frac{1}{p+2}|u|^{p+2} - \frac{1}{2}u^2$.
We have

$$E'(t) = fu' - cu'^2 \leq (\frac{p+2}{2pc})f^2 - \frac{(p+4)}{2(p+2)}cu'^2 \qquad \text{(By Young's inequality)}$$

and

$$(uu')' = u'^2 + uu'' = u'^2 + u(f - cu' - |u|^p u + u).$$

Hence

$$\begin{aligned}
\frac{d}{dt}(E + \frac{c}{p+2}uu') &\leq \frac{c}{p+2}u'^2 - \frac{(p+4)}{2(p+2)}cu'^2 - \frac{c}{p+2}uu' - \frac{c}{p+2}|u|^{p+2} + \frac{c}{p+2}u^2 \\
&\quad + \frac{c}{p+2}uf + \frac{f^2}{c}(\frac{p+2}{2p}) \\
&= -c(E + \frac{c}{p+2}uu') + \frac{c}{p+2}u^2 - \frac{c}{2}u^2 + \frac{c}{p+2}uf + \frac{f^2}{c}(\frac{p+2}{2p}) \\
&= -c(E + \frac{c}{p+2}uu') - \frac{pc}{2(p+2)}u^2 + \frac{f^2}{c}(\frac{p+2}{2p}) + \frac{c}{p+2}uf \\
&\leq -c(E + \frac{c}{p+2}uu') - \frac{pc}{2(p+2)}u^2 + \frac{pc}{2(p+2)}u^2 + \frac{c}{2p(p+2)}f^2 + \frac{f^2}{c}(\frac{p+2}{2p}) \\
&\leq -c(E + \frac{c}{p+2}uu') + \frac{1}{2p}(\frac{c}{p+2} + \frac{p+2}{c})f^2,
\end{aligned}$$

which provides

$$\frac{d}{dt}[\exp^{ct}(E + \frac{c}{p+2}uu')] \leq \frac{\exp^{ct}}{2p}(\frac{c}{p+2} + \frac{p+2}{c})f^2.$$

Integrating between t_0 and t we get

$$(E + \frac{c}{p+2}uu')(t) \leq \exp^{-c(t-t_0)}(E + \frac{c}{p+2}uu')(t_0) + \frac{c^2+(p+2)^2}{2pc^2(p+2)}\|f\|_\infty^2(1 - \exp^{-c(t-t_0)}).$$

Passing to the $\overline{\lim}$ we get

$$\overline{\lim_{t\to\infty}}(E(t)+\frac{c}{p+2}uu')\leq\frac{c^2+(p+2)^2}{2pc^2(p+2)}\|f\|_\infty^2.$$

For any $\varepsilon>0$ we have for t large enough

$$(E(t)+\frac{c}{p+2}uu')\leq\frac{c^2+(p+2)^2}{2pc^2(p+2)}\|f\|_\infty^2+\varepsilon.$$

Finally let us consider an "asymptotically maximizing" sequence t_n such that

$$\lim_{n\to\infty}u^2(t_n)=\overline{\lim_{t\to\infty}}\,u^2(t).$$

Assuming this limit to be positive, since u'' is bounded we have $\lim\limits_{n\to\infty}u'(t_n)=0$. In fact, if we suppose that $u'(t_n)\to a>0$, there exists an interval $]t_{n-d},t_{n+d}[$ with $d>0$ such that

$$u'(t_n)\leq\frac{a}{2}.$$

Then if

$$\overline{\lim_{t\to+\infty}}\,u(t_n)=M,$$

we have

$$\liminf u(t_{n+d})=M+\frac{ad}{2}.$$

This is impossible. We have the same resoning if $a<0$. Then $\lim_{n\to\infty}u'(t_n)=0$. Consequently for n large enough we have

$$\frac{1}{p+2}|u|^{p+2}(t_n)-\frac{1}{2}u^2(t_n)\leq E(t_n)+\frac{\varepsilon}{2}\leq\frac{c^2+(p+2)^2}{2pc^2(p+2)}\|f\|_\infty^2+2\varepsilon$$

$$\Rightarrow|u|^{p+2}(t_n)-\frac{p+2}{2}u^2(t_n)\leq\frac{c^2+(p+2)^2}{2pc^2}\|f\|_\infty^2+2(p+2)\varepsilon.$$

Using Young's inequality we have

$$\frac{p+2}{2}|u|^2\leq\frac{p}{p+2}(\frac{p+2}{2})^{\frac{p+2}{p}}+\frac{2}{p+2}|u|^{p+2},$$

hence for $t=t_n$

$$|u|^{p+2}\leq\frac{p}{p+2}(\frac{p+2}{2})^{\frac{p+2}{p}}+\frac{2}{p+2}|u|^{p+2}+\frac{c^2+(p+2)^2}{2pc^2}\|f\|_\infty^2+2(p+2)\varepsilon.$$

Then

$$\frac{p}{p+2}|u|^{p+2}\leq\frac{p}{p+2}(\frac{p+2}{2})^{\frac{p+2}{p}}+\frac{c^2+(p+2)^2}{2pc^2}\|f\|_\infty^2+2(p+2)\varepsilon,$$

and consequently

$$|u|^{p+2}(t) \leq (\frac{p+2}{2})^{\frac{p+2}{p}} + (p+2)[\frac{c^2+(p+2)^2}{2p^2c^2}]\|f\|_\infty^2 + 2\frac{(p+2)^2}{p}\varepsilon.$$

This implies

$$\overline{\lim_{t\to\infty}} |u|^{p+2}(t) \leq (\frac{p+2}{2})^{\frac{p+2}{2}} + (p+2)[\frac{c^2+(p+2)^2}{2p^2c^2}]\|f\|_\infty^2 + 2\frac{(p+2)^2}{p}\varepsilon,$$

and since ε is arbitrary we obtain (1.23).

6 A precise estimate for c large.

When $c \geq 2\sqrt{p}$, the inequality (1.23) and the positivity preserving property of $\mathcal{L}^{-1}$ allow a more precise estimate on u for t large.

Proposition 1.2. *For any $c \geq 2\sqrt{p}$, we have*

$$\overline{\lim_{t\to\infty}} |u(t)| \leq 1 + \frac{1}{p}\|f\|_\infty \tag{1.24}$$

valid whenever $\|f\|_\infty \leq 1$.

Proof. When $c \geq 2\sqrt{p}$, the operator $\mathcal{L}u = u'' + cu' + pu$ has positive inverse on L^∞ (see (1.4)). In addition the estimate (1.23) here provides

$$\overline{\lim_{t\to\infty}} |u|^{p+2}(t) \leq (\frac{p+2}{2})^{\frac{p+2}{p}} + (p+2)[\frac{4p+(p+2)^2}{8p^3}]\|f\|_\infty^2.$$

In particular if we assume

$$\|f\|_\infty \leq 1,$$

then we find

$$\overline{\lim_{t\to\infty}} |u|^{p+2}(t) \leq (\frac{p+2}{2})^{\frac{p+2}{p}} + (p+2)[\frac{4p+(p+2)^2}{8p^3}].$$

However for $p \geq 2$ we have $\frac{4p+(p+2)^2}{8p^3} \leq \frac{1}{2}$ and thus

$$\begin{aligned} |u|^{p+2} &\leq (\frac{p+2}{2})^{\frac{p+2}{p}} + \frac{p+2}{2} \\ &\leq 2(\frac{p+2}{2})^{\frac{p+2}{p}}. \end{aligned}$$

Hence

$$|u| \leq 2^{\frac{1}{p+2}}(\frac{p+2}{2})^{\frac{1}{p}}.$$

Now if u is any solution of (1.1) we set $u = 1 + v$ so we have

$$v'' + cv' + pv + [|v+1|^p(v+1) - ((p+1)v+1)] = f.$$

Our aim now is to prove that

$$v'' + cv' + pv \leq f.$$

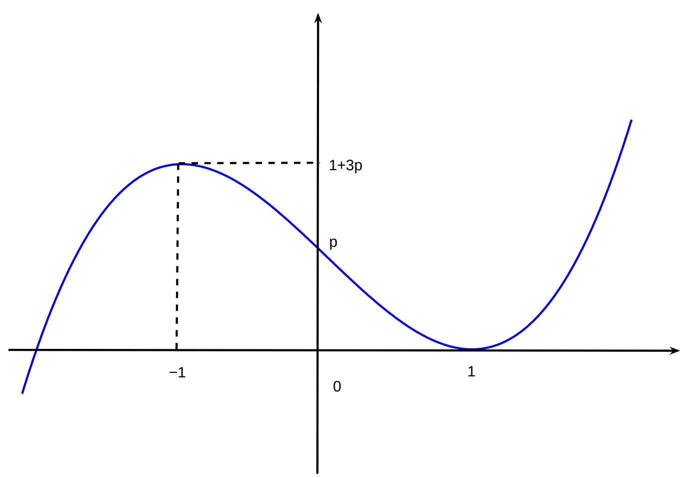
For this we should have that $|v+1|^p(v+1) - ((p+1)v+1) \geq 0$ for $|u| \leq 2^{\frac{1}{p+2}}(\frac{p+2}{2})^{\frac{1}{p}}$.
Let us set for $u \geq -2^{\frac{1}{p+2}}(\frac{p+2}{2})^{\frac{1}{p}}$

$$h(u) = |v+1|^p(v+1) - ((p+1)v+1) = |u|^p u - (p+1)u + p.$$

We have

$$h'(u) = (p+1)(|u|^p - 1),$$

and the shape of the representative curve of h is as follow:



According to the curve $h(u) \geq 0 \ \forall \ u \in [-1, +\infty[$.
on $]-\infty, -1]$; h is an increasing function then

$$h(u) \geq h(-2^{\frac{1}{p+2}}(\frac{p+2}{2})^{\frac{1}{p}}) \quad \forall u \geq -2^{\frac{1}{p+2}}(\frac{p+2}{2})^{\frac{1}{p}}$$

since $-2^{\frac{1}{p+2}}(\frac{p+2}{2})^{\frac{1}{p}} \in]-\infty, -1]$.
In another hand $2^{\frac{1}{p+2}}(\frac{p+2}{2})^{\frac{1}{p}} \leq (p+2)^{\frac{1}{p}}$, so that

$$h(-(p+2)^{\frac{1}{p}}) \leq h(-2^{\frac{1}{p+2}}(\frac{p+2}{2})^{\frac{1}{p}}).$$

However

$$h(-(p+2)^{\frac{1}{p}}) = -(p+2)^{\frac{1}{p}} + p \geq 0$$

since $p^p \geq p+2 \; \forall \; p \geq 2$.
Thus

$$h(u) \geq 0 \quad \forall \; u \geq -2^{\frac{1}{p+2}}(\frac{p+2}{2})^{\frac{1}{p}}.$$

In particular we have

$$v'' + cv' + pv \leq f.$$

We claim that

$$\overline{\lim_{t\to\infty}}\, u(t) \leq 1 + \frac{1}{p}\|f\|_\infty.$$

Assuming that this inequality is false, we can select $\delta > 0$ and t_n some sequence tending to $+\infty$ such that

$$u(t_n) \geq 1 + \frac{1}{p}\|f\|_\infty + \delta.$$

Now replace v by $v(t+t_n)$ and f by $f(t+t_n)$. We can pass to the limit along a subsequence, for which the sequence of the shifted of f converges in L^2 weak. We can also assume that the shifted of v converge in C^1, then the limiting fuction fulfills the limiting equation. Finally we are reduced to consider the case where v is bounded on $\mathbb{R}$. In this case since

$$v'' + cv' + pv \leq \|f\|_\infty,$$

we obtain

$$v \leq \frac{1}{p}\|f\|_\infty$$

in contradiction with

$$v(t_n) \geq \frac{1}{p}\|f\|_\infty + \delta,$$

so that

$$\overline{\lim_{t\to\infty}}\, u(t) \leq 1 + \frac{1}{p}\|f\|_\infty.$$

Finally, we obtain an analogue inequality by changing u to -u, and the result follows.

Remark 1.2. *The result of Proposition* 1.2 *is no longer true for small values of* c *even for* $p = 2$, *[11].*

7 Proof of Theorem 1.5 and its corollaries.

One of the main ingredients of the proof is a precise formulation of the asymptotic stability of the bounded solution ω_+, ω_-. Of course, changing u and f to their opposites it is sufficent to consider ω_+. In this case

Lemma 1.1. *Assume that*

$$\|f\|_\infty < \frac{c}{\sqrt{1+c^2}} \frac{\sqrt{3p-4}}{8\sqrt{3p}}. \tag{1.25}$$

Then for any $\delta < \frac{1}{2\sqrt{6p}}$*, there exists* $\eta > 0$ *such that the conditions*

$$|u(t_0) - 1| \le \delta \ et \ |u'(t_0)| \le \eta$$

imply

$$\forall t \ge t_0 \ , \ |u(t) - 1| \le \frac{1}{2p} \tag{1.26}$$

and

$$\overline{\lim_{t\to+\infty}} |u(t) - 1| \le \frac{\sqrt{1+c^2}}{c} \frac{2\sqrt{6}}{\sqrt{3p-4}} \ \ \|f\|_\infty. \tag{1.27}$$

In addition if $c \ge 2\sqrt{p}$*, under the same assumptions we have*

$$\lim_{t\to+\infty} (|u(t) - \omega_+(t)| + |u'(t) - \omega_+'(t)|) = 0. \tag{1.28}$$

Proof: By setting $u = 1 + v$ we obtain the equation for v

$$v'' + cv' + |1+v|^p(1+v) - (v+1) = f.$$

Let

$$P(v) = \frac{(v+1)^{p+2}}{p+2} - \frac{1}{2}(v+1)^2 + \frac{1}{2} - \frac{1}{p+2},$$

there exists $\theta \in]0,1[$ such that

$$P(v) = \frac{p}{2}v^2 + \frac{p(p+1)}{6}(1+\theta v)^{p-1}v^3.$$

We remark that

$$|v| \le \frac{1}{2p} \Longrightarrow \frac{5p}{16}v^2 \le P(v) \le \frac{11p}{16}v^2.$$

In fact

$$|P(v) - \frac{p}{2}v^2| \le \frac{p+1}{12}(1+\theta v)^{p-1}v^2 \quad if\ |v| \le \frac{1}{2p}.$$

However

$$(p+1)(1+\theta v)^{p-1} \le (p+1)(1+\frac{1}{2p})^{p-1} = p(1+\frac{1}{p})(1+\frac{1}{2p})^{p-1}$$
$$\le \sqrt{e}p(1+\frac{1}{2p}) \le \frac{9}{5}p(1+\frac{1}{4})$$

so that

$$|P(v) - \frac{p}{2}v^2| \le \frac{3p}{16}v^2.$$

We now introduce

$$F(t) = \frac{1}{2}v'^2(t) + P(v)(t)$$

and

$$\Phi(t) = F(t) + \alpha vv'(t)$$

where $\alpha > 0$ will be chosen later. First we notice that if $\alpha < \frac{1}{4}$, we have

$$|\alpha vv'| \le \frac{1}{8}v'^2 + \frac{p}{16}v^2,$$

so that

$$\frac{1}{4}(pv^2 + v'^2) \le \Phi(t) \le \frac{3}{4}(pv^2 + v'^2)$$

whenever the condition

$$|v| \le \frac{1}{2p}$$

is fulfilled.
Let

$$T = \sup\{t \ge t_0,\ |v(t)| \le \frac{1}{2p}\}$$

and $J := [t_0, T)$. We now derive a sequence of estimates valid for $t \in J$. We have

$$F'(t) = -cv'^2 + fv' \le -\frac{c}{2}v'^2 + \frac{f^2}{2c}$$

$$(vv')' = v'^2 + vv'' = v'^2 - cvv' - vg(v) + fv$$

with $g(v) = |1+v|^p(1+v) - (1+v) = pv + \frac{p(p+1)}{2}(1+\theta v)^{p-1}v^2$.

- If $v \ge 0$ then $g(v)v \ge pv^2$.
- If $v \le 0$, we have $(1+\theta v)^{p-1} \le 1$ then $g(v)v \ge v^2(\frac{3p}{4} - \frac{1}{4})$.

So that

$$(vv')' \le v'^2 - cvv' - v^2(\frac{3p}{4} - \frac{1}{2}) + f^2.$$

By using
$$-cvv' \leq \frac{1}{2}v^2 + \frac{c^2}{2}v'^2,$$
we deduce
$$\Phi' \leq (\alpha(1+\frac{c^2}{2}) - \frac{c}{2})v'^2 - \alpha(\frac{3p}{4} - 1)v^2 + (\alpha + \frac{1}{2c})f^2.$$
We select
$$\alpha = \frac{c}{4(1+\frac{c^2}{2})}$$
so that $\alpha(1+\frac{c^2}{2}) = \frac{c}{4}$ and
$$\begin{aligned}\Phi' &\leq \frac{c}{4}v'^2 - \alpha(\frac{3p}{4} - 1)v^2 + (\alpha + \frac{1}{2c})f^2 \\ &\leq -\frac{\alpha(\frac{3p}{4}-1)}{p}(pv^2 + v'^2) + (\alpha + \frac{1}{2c})f^2.\end{aligned}$$
Therefore we find
$$\Phi' \leq \frac{-4\alpha(\frac{3p}{4}-1)}{3p}\Phi + (\alpha + \frac{1}{2c})f^2,$$
which is easily integrated to give
$$\forall t \in J, \;\; \Phi(t) \leq \exp[-4\alpha(\frac{\frac{3p}{4}-1}{3p})(t-t_0)]\Phi(t_0) + (\frac{6p}{3p-4})(\frac{1+c^2}{c^2})\|f\|_\infty^2.$$
In order to achieve $T = \infty$, we need to ensure $|v| < \frac{1}{2p}$ on J, which is satisfied as soon as
$$\Phi(t_0) + (\frac{6p}{3p-4})(\frac{1+c^2}{c^2})\|f\|_\infty^2 \leq \frac{p}{4}(\frac{1}{2p})^2 = \frac{1}{16p}.$$
To achieve this condition it is sufficent to have
$$\Phi(t_0) \leq \frac{1}{32p} \quad \text{et} \quad (\frac{6p}{3p-4})(\frac{1+c^2}{c^2})\|f\|_\infty^2 \leq \frac{1}{32p}.$$
The first inequality is satisfied whenever $\frac{3}{4}pv^2(t_0) < \frac{1}{32p}$ et $\frac{3}{4}v'^2(t_0) \leq \frac{1}{32p} - \frac{3}{4}v^2(t)$ which corresponds to our hypothesis. The second condition is equivalent to
$$\|f\|_\infty < \frac{c}{\sqrt{1+c^2}}\frac{\sqrt{3p-4}}{8\sqrt{3p}}$$
for p=2 it is reduced to
$$\|f\|_\infty \leq \frac{c}{\sqrt{1+c^2}}\frac{\sqrt{2}}{16\sqrt{3}}.$$

Under this conditions we have $T = \infty$ and

$$\forall t \geq t_0,\ \Phi(t) \leq \exp[-4\alpha(\frac{\frac{3p}{4}-1}{3p})t]\Phi(t_0) + (\frac{6p}{3p-4})(\frac{1+c^2}{c^2})\|f\|_\infty^2.$$

Moreover the inequality

$$|u(t)-1| \leq \frac{2}{\sqrt{p}}\Phi(t)^{\frac{1}{2}}$$

gives that

$$\overline{\lim_{t\to+\infty}}\,|u(t)-1| \leq \frac{\sqrt{1+c^2}}{c}\frac{2\sqrt{6}}{\sqrt{3p-4}}\ \ \|f\|_X.$$

To prove the second part, we observe that the asymptotic distance between u and 1 is less than

$$\frac{1}{2p} < (\frac{2p+1}{p+1})^{\frac{1}{p}} - 1 = r_0.$$

In fact $\frac{2p+1}{p+1} = 1 + \frac{p}{p+1} \geq \frac{5}{3}$ and $(1+\frac{1}{2p})^p < \sqrt{e} < \frac{5}{3}$.

Then we claim that u approches ω_+ as t tends to infinity and since u'' is bounded it will follow that $u' \to \omega'_+$. To prove this we now use the translation-compactness method developed by Amerio [2] and Biroli [3]. Assuming, by contradiction, the existence of α_n tending to infinity with

$$\lim_{n\to\infty} |u(\alpha_n) - \omega_+(\alpha_n)| = \eta > 0$$

we can replace α_n by a sequence still denoted α_n for convenience, such that

$$u(\alpha_n+t),\ \ \omega_+(\alpha_n+t),\ \ f(\alpha_n+t)$$

converge respectively to v, ω and g on $\mathbb{R}$, uniformly on compacts for the first two functions in local L^2 weak for the third. Then v and ω are two bounded solutions of

$$z'' + cz' + |z|^p z - z = g$$

with

$$Max\{\|v-1\|_\infty, \|\omega-1\|_\infty\} < r_0.$$

In particular $v = \omega$ and for $t = 0$ we obtain a contradiction with

$$\lim_{n\to\infty} |u(\alpha_n) - \omega_+(\alpha_n)| = \eta > 0.$$

This contradiction proves the claim and complete the proof of lemma 1.1.

Lemma 1.2. *([11], lemma 6.2) Let $J = (a, +\infty)$ and $u \in C^2(J)$ such that $u \leq M$ on J. Let*

$$U := \overline{\lim_{t\to+\infty}}\, u(t)$$

Then there exists a sequence of reals $t_n \in J$ such that $t_n \to +\infty$ and

$$\overline{\lim_{n\to+\infty}}\, u''(t_n) \leq 0, \quad \lim_{n\to+\infty} u(t_n) = U$$

Lemma 1.3. *For any $\varepsilon > 0$, the inequality $u - |u|^p u \leq \varepsilon$ implies either : $u \leq \frac{p+1}{p}\varepsilon$ or $u \geq 1 - \sqrt{3}\varepsilon$.*

Proof. If $u \leq 0$ there is nothing to prove. If $u > 0$ we distinguish 2 cases
i) If $u \leq \frac{1}{(p+1)^{\frac{1}{p}}}$, then $1 - |u|^p \geq \frac{p}{p+1}$ and therefore

$$u - |u|^p u = u(1 - |u|^p) \leq \varepsilon \Rightarrow \frac{p}{p+1} u \leq \varepsilon \Rightarrow u \leq \frac{p+1}{p}\varepsilon.$$

ii) If $u \geq \frac{1}{(p+1)^{\frac{1}{p}}}$, then $u(1-u^p) = u(1-u)[\frac{1-u^p}{1-u}] \geq \frac{1}{(p+1)^{\frac{1}{p}}}(1-u)$ then since $(p+1)^{\frac{1}{p}} \leq \sqrt{3}$ we have $u \geq 1 - \sqrt{3}\varepsilon$.

Proof of Theorem 1.5**.**

Let u be a solution of (1.1) on $\mathbb{R}$ and introduce

$$M = \overline{\lim_{t\to+\infty}}\, u(t), \quad m = \underline{\lim}_{t\to+\infty}\, u(t), \quad \varepsilon = \|f\|_\infty.$$

As a consequence of lemma 1.2, there exists a sequence of reals t_n such that

$$\lim_{n\to+\infty} u''(t_n) \leq 0, \quad \lim_{n\to\infty} u(t_n) = M.$$

Since u'' is bounded $\Rightarrow \lim_{n\to\infty} u'(t_n) = 0$. Now we have

$$(u - |u|^p u)(t_n) = -f(t_n) + u''(t_n) + cu'(t_n)$$

and therefore

$$\overline{\lim_{n\to\infty}}\, (u - |u|^p u)(t_n) \leq \varepsilon.$$

As a consequence of lemma 1.3, for n large enough we have either

$$u(t_n) \leq \frac{p+1}{p}\varepsilon < 2\varepsilon$$

or

$$u(t_n) \geq 1 - \sqrt{3}\varepsilon.$$

In the first case we conclude

$$M \leq 2\varepsilon.$$

In the second case we have in fact, as a consequence of section 5

$$1 - \sqrt{3}\varepsilon \leq u(t_n) \leq 1 + \frac{\varepsilon}{p}.$$

As a consequence of lemma 1.1, since (1.15) implies

$$\sqrt{3}\varepsilon \leq \frac{1}{2p\sqrt{6}} \text{ and since } \lim_{n\to\infty} u'(t_n) = 0,$$

we conclude that u is approching ω_+ at $+\infty$. In this case the proof is over. Coming back to the first case, we now consider a sequence s_n such that

$$u''(s_n) \geq 0, \quad \lim_{n\to\infty} u(s_n) = m$$

and by the same argument as above we conclude that either u approches ω_- at $+\infty$, or

$$m \geq -2\varepsilon.$$

In this second and last case we have

$$\overline{\lim_{t\to+\infty}} |u(t)| \leq 2\varepsilon$$

and by hypothesis on f this implies

$$\lim_{t\to\infty} |u(t) - \omega_0(t)| = 0.$$

We prove this last property using again the translation method of Amerio-Biroli . Indeed assuming, on the contrary, the existence of α_n tending to infinity with

$$\lim_{n\to\infty} |u(\alpha_n) - \omega_0(\alpha_n)| = \eta > 0,$$

we can replace α_n by a sequence, still denoted α_n for convenience, such that

$$u(\alpha_n + t), \;\; \omega_0(\alpha_n + t), \;\; f(\alpha_n + t)$$

converge respectivly to v, ω et g on $\mathbb{R}$, uniformly on compacts for the first two functions, in local L^2 weak for the third. Then v, ω are two bounded solutions of

$$z'' + cz' + |z|^p z - z = g$$

with

$$Max\{\|v\|_\infty, \|\omega\|_\infty\} \leq 2\varepsilon.$$

In particular $v = \omega$ and for $t = 0$ we obtain a contradiction with

$$\lim_{n\to\infty} |u(\alpha_n) - \omega_0(\alpha_n)| = \eta > 0.$$

This contradiction proves the claim and completes the proof of theorem 1.5.

8 Proof of Theorem 1.6.

First we show that for a fixed period T , T-periodic solutions u are such that $|||u|^p u - u||_\infty$ tends to 0 with $||f||_\infty$.

Proposition 1.3. *Let f be bounded, T-periodic and let $u \in C^2(\mathbb{R})$ be a T-periodic solution of* (1.1)*. Then we have the estimate*

$$\||u|^p u - u\|_\infty \le \|f\|_\infty (1 + \frac{\sqrt{T}}{c}\sqrt{K\|f\|_\infty^p + (\frac{p^4(p+1)}{(p^2-p-1)} + 1)T}).$$

with $K = 2^{(1+\frac{p}{2}(p-3)^+)} \frac{p^{\frac{p}{2}}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p-2}{2}}(p+1)T}{(p^2-p+3)^{\frac{p-2}{2}}c^p} + 2^{(p-3)^+}p^2(p^2-1)\frac{T^{p-1}}{c^p}$
$+ p^p(p+1)T$.

Proof. By integrating (1.1) on $J = (0, T)$ we find

$$\int_J (|u|^p u - u)dt = \int_J f dt,$$

in particular

$$|\frac{1}{T}\int_J (|u|^p u - u)dt| = ||f||_\infty.$$

Then multiplying (1.1) by u' and integrating on J we get

$$c\int_J u'^2 dt = \int_J f u' dt \Rightarrow c\int_J u'^2 dt \le (\int_J f^2 dt)^{\frac{1}{2}}(\int_J u'^2 dt)^{\frac{1}{2}},$$

hence

$$||u'||_2 \le \frac{\sqrt{T}}{c}||f||_\infty. \tag{1.29}$$

Next multiplying (1.1) by $u|u|^{p-2}$ and integrating on J we obtain

$$\begin{aligned} &\int_J (|u|^{2p} - |u|^p)dt = \int_J f u|u|^{p-2}dt + (p-1)\int_J u'^2|u|^{p-2}dt \\ \Longrightarrow &\int_J (|u|^{2p} - |u|^p)dt \le ||f||_\infty \int_J |u|^{p-1}dt + (p-1)\|u\|_\infty^{p-2}dt\int_J u'^2 dt. \end{aligned} \tag{1.30}$$

By young's inequality we have

$$||f||_\infty |u|^{p-1} \le \frac{p-1}{p^2}|u|^p + p^{p-2}\|f\|_\infty^p,$$

hence

$$||f||_\infty \int_J |u|^{p-1}dt \le \frac{p-1}{p^2}\int_J |u|^p dt + p^{p-2}\|f\|_\infty^p T. \tag{1.31}$$

On the other hand we have

$$\|u\|_\infty^{p-2} \le 2^{(p-3)^+} T^{\frac{2-p}{p}} \|u\|_p^{p-2} + 2^{(p-3)^+} T^{\frac{p-2}{2}} \|u'\|_2^{p-2}, \tag{1.32}$$

where

$$(p-3)^+ = \begin{cases} p-3 & \text{si} \quad p \ge 3, \\ 0 & \text{si} \quad 2 \le p < 3. \end{cases}$$

In fact

$$\|u\|_\infty \le \frac{1}{T}\|u\|_1 + \|u'\|_1,$$

using Hölder's inequality for $\|u\|_1$ and $\|u'\|_1$ we have

$$\|u\|_1 \le T^{\frac{p-1}{p}} \|u\|_p \quad et \quad \|u'\|_1 \le T^{\frac{1}{2}} \|u'\|_2.$$

If $2 \le p < 3$ we have

$$\|u\|_\infty^{p-2} \le T^{\frac{2-p}{p}} \|u\|_p^{p-2} + T^{\frac{p-2}{2}} \|u'\|_2^{p-2},$$

and if $p \ge 3$ the function x^{p-2} is convex hence

$$\|u\|_\infty^{p-2} \le 2^{p-3} T^{\frac{2-p}{p}} \|u\|_p^{p-2} + 2^{p-3} T^{\frac{p-2}{2}} \|u'\|_2^{p-2}.$$

Applying Young's inequality we have

$$\begin{aligned} \|u\|_p^{p-2} \le & \frac{(p^2-p+3)c^2}{2^{(p-3)^+} p^2 (p-1) T^{\frac{2}{p}} \|f\|_\infty^2} \int_J |u|^p \\ & + 2^{(\frac{(p-2)(p-3)^+}{2}+1)} [\frac{p(p-1)(p-2)}{p^2-p+3}]^{\frac{p-2}{2}} \frac{T^{\frac{p-2}{p}}}{pc^{p-2}} \|f\|_\infty^{p-2}. \end{aligned} \tag{1.33}$$

In fact

$$\lambda \|u\|_p^{p-2} \times \frac{1}{\lambda} \le \frac{\lambda^r}{r} \|u\|_p^{(p-2)^r} + \frac{1}{r' \lambda^{r'}}$$

with

$$r = \frac{p}{p-2} \;\; ; \;\; r' = \frac{p}{2},$$

and

$$\frac{\lambda^r}{r} = \frac{(p^2-p+3)c^2}{2^{(p-3)^+} p^2 (p-1) T^{\frac{2}{p}} \|f\|_\infty^2}.$$

Hence by (1.29), (1.32) and (1.33) we have

$$\begin{aligned} (p-1)\|u\|_\infty^{p-2} \int_J u'^2 dt \le & \frac{p^2-p+3}{p^2} \int_J |u|^p \; + \; 2^{(p-3)^+} (p-1) \frac{T^{p-1}}{c^p} \|f\|_\infty^p \\ & + 2^{(1+\frac{p}{2}(p-3)^+)} \frac{p^{\frac{p-4}{2}} (p-1)^{\frac{p}{2}} (p-2)^{\frac{p-2}{2}} T}{(p^2-p+3)^{\frac{p-2}{2}} c^p} \|f\|_\infty^p. \end{aligned} \tag{1.34}$$

Remplacing (1.31) and (1.34) in (1.30) we get

$$\begin{aligned}\int_J (|u|^{2p} - |u|^p)dt \leq & \frac{p^2+2}{p^2}\int_J |u|^p + p^{p-2}T\|f\|_\infty^p + 2^{(p-3)^+}(p-1)\frac{T^{p-1}}{c^p}\|f\|_\infty^p \\ & + 2^{(1+\frac{p}{2}(p-3)^+)}\frac{p^{\frac{p-4}{2}}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p-2}{2}}T}{(p^2-p+3)^{\frac{p-2}{2}}c^p}\|f\|_\infty^p,\end{aligned}$$

which implies

$$\begin{aligned}\int_J (|u|^{2p} - \frac{2}{p^2}|u|^p)dt \leq & 2\int_J |u|^p + p^{p-2}T\|f\|_\infty^p + 2^{(p-3)^+}(p-1)\frac{T^{p-1}}{c^p}\|f\|_\infty^p \\ & + 2^{(1+\frac{p}{2}(p-3)^+)}\frac{p^{\frac{p-4}{2}}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p-2}{2}}T}{(p^2-p+3)^{\frac{p-2}{2}}c^p}\|f\|_\infty^p.\end{aligned}$$

Using Young's inequality we have

$$|u|^p \leq \frac{p^2-p-1}{2p^2}|u|^{2p} + \frac{p^2}{2(p^2-p-1)}.$$

In fact

$$|u|^p \leq \frac{{\lambda_1}^r}{r}|u|^{pr} + \frac{1}{r'{\lambda_1}^{r'}},$$

with

$$r = r' = 2$$

and

$$\frac{{\lambda_1}^r}{r} = \frac{p^2-p-1}{2p^2}.$$

Consequently we have

$$\begin{aligned}\int_J (|u|^{2p} - \frac{2}{p^2}|u|^p)dt \leq & \frac{p^2-p-1}{p^2}\int_J |u|^{2p} + \frac{p^2}{(p^2-p-1)}T \\ & + p^{p-2}T\|f\|_\infty^p + 2^{(p-3)+}(p-1)\frac{T^{p-1}}{c^p}\|f\|_\infty^p \\ & + 2^{(1+\frac{p}{2}(p-3)^+)}\frac{p^{\frac{p-4}{2}}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p-2}{2}}T}{(p^2-p+3)^{\frac{p-2}{2}}c^p}\|f\|_\infty^p.\end{aligned}$$

Hence

$$\begin{aligned}\int_J (\frac{(p+1)}{p^2}|u|^{2p} - \frac{2}{p^2}|u|^p)dt \leq & \frac{p^2}{(p^2-p-1)}T + 2^{(p-3)^+}(p-1)\frac{T^{p-1}}{c^p}\|f\|_\infty^p + p^{p-2}T\|f\|_\infty^p \\ & + 2^{(1+\frac{p}{2}(p-3)^+)}\frac{p^{\frac{p-4}{2}}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p-2}{2}}T}{(p^2-p+3)^{\frac{p-2}{2}}c^p}\|f\|_\infty^p.\end{aligned}$$

Multiplying this inequality by $(p+1)p^2$ we have

$$\begin{aligned}(p+1)^2\int_J |u|^{2p}dt - 2(p+1)\int_J |u|^p dt \leq & 2^{(1+\frac{p}{2}(p-3)^+)}\frac{p^{\frac{p}{2}}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p-2}{2}}(p+1)T}{(p^2-p+3)^{\frac{p-2}{2}}c^p}\|f\|_\infty^p \\ & + 2^{(p-3)^+}p^2(p-1)(p+1)\frac{T^{p-1}}{c^p}\|f\|_\infty^p + p^p(p+1)T\|f\|_\infty^p \\ & + \frac{p^4(p+1)}{(p^2-p-1)}T.\end{aligned}$$

Adding T we get

$$\begin{aligned}\|(p+1)|u|^p - 1\|_2^2 \leq & 2^{(1+\frac{p}{2}(p-3)^+)}\frac{p^{\frac{p}{2}}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p-2}{2}}(p+1)T}{(p^2-p+3)^{\frac{p-2}{2}}c^p}\|f\|_\infty^p \\ & + 2^{(p-3)^+}p^2(p^2-1)\frac{T^{p-1}}{c^p}\|f\|_\infty^p + p^p(p+1)T\|f\|_\infty^p \\ & + (\frac{p^4(p+1)}{(p^2-p-1)}+1)T.\end{aligned}$$

Finally we find

$$\begin{aligned}\||u|^p u - u\|_\infty & \leq \|f\|_\infty + \|(|u|^p u - u)'\|_1 \\ & \leq \|f\|_\infty + \|(p+1)|u|^p - 1\|_2 \|u'\|_2 \\ & \leq \|f\|_\infty (1 + \frac{\sqrt{T}}{c}\sqrt{K\|f\|_\infty^p + (\frac{p^4(p+1)}{(p^2-p-1)}+1)T}),\end{aligned}$$

with $K = 2^{(1+\frac{p}{2}(p-3)^+)}\frac{p^{\frac{p}{2}}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p-2}{2}}(p+1)T}{(p^2-p+3)^{\frac{p-2}{2}}c^p} + 2^{(p-3)^+}p^2(p^2-1)\frac{T^{p-1}}{c^p}$
$+ p^p(p+1)T$.

In order to prove Theorem 1.6, the following simple lemma is useful.

Lemma 1.4. *For any $\varepsilon > 0$, the inequality $||u|^p u - u| \leq \varepsilon$ implies*

$$\inf\{|u|, |1-u|, |1+u|\} \leq (\frac{p+1}{p})\varepsilon$$

Proof. i) If $|u| < \frac{1}{(p+1)^{\frac{1}{p}}}$, then $1 - |u|^p > \frac{p}{p+1}$ and therefore
$||u|^p - u| = |u||1-|u|^p| \leq \varepsilon \Longrightarrow \frac{p}{p+1}|u| \leq \varepsilon \Longrightarrow |u| \leq \frac{p+1}{p}\varepsilon$.
ii)If $|u| \geq \frac{1}{(p+1)^{\frac{1}{p}}}$, then $|u||1-|u|^p| = |u(1-u)||\frac{1-|u|^p}{1-u}| \geq \frac{p|1-u|}{(p+1)[(p+1)^{\frac{1}{p}}-1]}$ and therefore

$$u \geq 1 - \varepsilon\frac{(p+1)}{p}[(p+1)^{\frac{1}{p}} - 1]$$

$$\Longrightarrow |1-|u|| \leq \varepsilon(\frac{p+1}{p}).$$

The result follows since

$$|1-|u|| = \inf\{|1-u|, |1+u|\}.$$

Proof of Theorem 1.6.
Under the hypothesis (1.16), as a consequence of proposition 1.3 and lemma 1.4 , any T-periodic solution u of (1.1) satisfies, for each t,

$$\inf\{|u(t)|, |1-u(t)|, |1+u(t)|\} \leq \inf\{(1+\frac{c\sqrt{p}}{2(p+1)})^{\frac{1}{p}}-1, (\frac{2p+1}{p+1})^{\frac{1}{p}}-1\}.$$

Since u is continuous and the 3 closed intervals centered at $0, 1, -1$ with radius

$$\rho = \inf\{(1+\frac{c\sqrt{p}}{2(p+1)})^{\frac{1}{p}}-1, (\frac{2p+1}{p+1})^{\frac{1}{p}}-1\}$$

are disjoint, we have either

$$\|u-1\|_\infty \leq \inf\{(1+\frac{c\sqrt{p}}{2(p+1)})^{\frac{1}{p}}-1, (\frac{2p+1}{p+1})^{\frac{1}{p}}-1\},$$

in wich case $u = \omega_+$, or

$$\|u+1\|_\infty \leq \inf\{(1+\frac{c\sqrt{p}}{2(p+1)})^{\frac{1}{p}}-1, (\frac{2p+1}{p+1})^{\frac{1}{p}}-1\},$$

in wich case $u = \omega_-$, or

$$\|u\|_\infty \leq \inf\{(1+\frac{c\sqrt{p}}{2(p+1)})^{\frac{1}{p}}-1, (\frac{2p+1}{p+1})^{\frac{1}{p}}-1\} < \frac{1}{(p+1)^{\frac{1}{p}}},$$

in wich case $u = \omega_0$. $\diamond$

Chapter 2

Boundedness for strongly damped and forced single well Duffing's equation

1 Introduction

We consider the second order ODE

$$u'' + cu' + g(u) = f(t), \tag{2.1}$$

where $c > 0$, $f \in L^\infty([t_0, +\infty))$ and $g \in C^1(\mathbb{R})$ satisfies some sign hypotheses. The typical case is

$$g(u) = bu + a|u|^p u. \tag{2.2}$$

More generally we shall assume that $g(0) = 0$ and for some $b > 0$

$$\forall u \in \mathbb{R}, \int_0^u g(s)ds, \int_0^u sg'(s)ds \geq \frac{b}{2}u^2. \tag{2.3}$$

A. Haraux [9] established that if

$$\forall s \in \mathbb{R}, \quad g'(s) \geq b,$$

then all solutions of (2.1) are ultimately bounded and more precisely:
• If $c \leq 2\sqrt{b}$, then

$$\overline{\lim_{t\to+\infty}} |u(t)| \leq \frac{2}{c\sqrt{b}} \overline{\lim_{t\to\infty}} |f(t)|$$

and

$$\overline{\lim_{t\to+\infty}} |u'(t)| \leq \left(\frac{2}{c} + \frac{1}{\sqrt{b}}\right) \overline{\lim_{t\to\infty}} |f(t)|.$$

• If $c \geq 2\sqrt{b}$ and in addition

$$\forall s \in \mathbb{R}, \quad g(s)s \geq 2\int_0^s g(u)du, \tag{2.4}$$

then

$$\overline{\lim_{t\to+\infty}} |u(t)| \leq \frac{1}{b} \overline{\lim_{t\to\infty}} |f(t)| \tag{2.5}$$

and

$$\overline{\lim_{t\to+\infty}} |u'(t)| \leq \frac{c}{b} \overline{\lim_{t\to\infty}} |f(t)|, \tag{2.6}$$

with

$$\overline{\lim_{t\to+\infty}} |f(t)| := \inf_T \|f\|_{L^\infty([T,+\infty))} = \lim_{T\to+\infty} \|f\|_{L^\infty([T,+\infty))}.$$

In this chapter, we prove the same result for the strongly damped case $c \geq 2\sqrt{b}$ without using the additional relaxed convexity hypothesis (2.4) and we improve the estimate (2.6).

2 Ultimate bound for c large

Theorem 2.1. *under the condition* (2.3) *and with*

$$c \geq 2\sqrt{b},$$

any solution of equation (2.1) *on* $J = [t_0, +\infty)$ *satisfies the estimates* (2.5) *and*

$$\overline{\lim_{t\to\infty}} |u'(t)| \leq \Big(\frac{1}{\sqrt{b}} + \frac{c - \sqrt{c^2 - 4b}}{2b}\Big) \overline{\lim_{t\to\infty}} |f(t)|. \tag{2.7}$$

Proof. Let

$$G(s) = \int_0^s g(u)du.$$

Introducing for all $t \in J$,

$$\Phi(t) := (u'^2 + 2G(u) + \alpha uu')(t),$$

where $\alpha = c - \sqrt{c^2 - 4b}$ we have

$$\Phi'(t) = u'^2(\alpha - 2c) - \alpha ug(u) - c\alpha uu' + f(2u' + \alpha u).$$

By an integration by parts of $\int_0^u sg'(s)ds$ and using (2.3) we get

$$g(u)u \geq G(u) + \frac{b}{2}u^2.$$

Then we have

$$\Phi'(t) \leq (\alpha - 2c)u'^2 - \alpha G(u) - \alpha\frac{b}{2}u^2 - c\alpha uu' + f(2u' + \alpha u).$$

Hence

$$\Phi'(t) \leq -\frac{\alpha}{2}(u'^2 + 2G(u) + \alpha uu') - \alpha\frac{b}{2}u^2 + (-c\alpha + \frac{\alpha^2}{2})uu' + (\frac{3\alpha}{2} - 2c)u'^2 + f(2u' + \alpha u).$$

On the other hand using Young's inequality we have

$$f(2u' + \alpha u) \leq \frac{\alpha}{2b}\|f\|_\infty^2 + \frac{b}{2\alpha}(4u'^2 + \alpha^2 u^2 + 4\alpha uu'),$$

where $\|f\|_\infty$ stands for $\|f\|_{L_\infty(J)}$, therefore

$$\Phi'(t) \leq -\frac{\alpha}{2}\Phi(t) + (-c\alpha + \frac{\alpha^2}{2} + 2b)uu' + (\frac{3\alpha}{2} - 2c + \frac{2b}{\alpha})u'^2 + \frac{\alpha}{2b}\|f\|_\infty^2.$$

We have $-c\alpha + \frac{\alpha^2}{2} + 2b = 0$ since $c - \sqrt{c^2 - 4b}$ is a solution of the equation $-cx + \frac{x^2}{2} + 2b = 0$. In addition we have

$$\frac{3\alpha}{2} - 2c + \frac{2b}{\alpha} = \alpha - c + \frac{\alpha}{2} - c + \frac{2b}{\alpha} = \alpha - c < 0.$$

Hence

$$\Phi'(t) \leq -\frac{\alpha}{2}\Phi(t) + \frac{\alpha}{2b}\|f\|_\infty^2$$

then

$$(\exp^{\frac{\alpha}{2}t}\Phi(t))' \leq \exp^{\frac{\alpha}{2}t}\frac{\alpha}{2b}\|f\|_\infty^2.$$

Integrating between t_0 and t we get

$$\Phi(t) \leq \exp^{-(t-t_0)}\Phi(t_0) + \frac{1}{b}\|f\|_\infty^2(1 - \exp^{-(t-t_0)}).$$

Passing to the $\overline{\lim}$ we get

$$\overline{\lim_{t\to\infty}}|\Phi(t)| \leq \frac{1}{b}\|f\|_\infty^2.$$

Hence we find the estimate

$$\overline{\lim_{t\to+\infty}}\Phi(t) \leq \frac{1}{b}F^2,$$

where $F = \overline{\lim}_{t\to+\infty}|f(t)|$.
In particular for any $\epsilon > 0$ and using condition (2.3), we have for t large enough

$$\alpha uu' + bu^2 \leq \Phi(t) \leq \frac{1}{b}F^2 + \epsilon.$$

Solving this differential inequality for u^2 we deduce

$$u^2 \leq \frac{1}{b}\big(\frac{1}{b}F^2 + 2\epsilon\big)$$

for $t \geq T(\epsilon)$. By letting $\epsilon \to 0$ we obtain (2.5). For the proof of (2.7) we notice that

$$\Phi(t) \geq u'^2 + \alpha u u' + b u^2 \geq \big(u' + \frac{\alpha}{2}u\big)^2.$$

Hence for t large enough we have

$$|u' + \frac{\alpha}{2}u| \leq \frac{1}{\sqrt{b}}F + \epsilon.$$

Then using (2.5) we obtain (2.7).

Remark 2.1. *When $c \geq 2\sqrt{b}$ we have*

$$\frac{1}{\sqrt{b}} + \frac{c - \sqrt{c^2 - 4b}}{2b} \leq \frac{c}{b}$$

therefore (2.7) *improves the estimate* (2.6) *but we do not recover the estimate given by W.S. Loud [16] which is $\frac{4}{c}$.*

$\diamond$

Chapter 3

Sharp estimates of bounded solutions to some second order equation in large damping case

1 Introduction

This chapter is a natural extension of the second one, in fact we use exactly the same calculation to obtain an estimate for the L^∞ norm of the solutions of some second order evolution equation in the case of strong dissipation.
Let H be a real Hilbert space. In the sequel we denote by (u, v) the inner product of two vectors u, v in H and by $|u|$ the H-norm of u.
Let $A : D(A) \to H$ a possibly unbounded self-adjoint linear operator such that

$$\exists \lambda > 0, \forall u \in D(A), \quad (Au, u) \geq \lambda |u|^2.$$

We consider the largest possible number satisfying the above inequality, in other words

$$\lambda_1 = \inf_{u \in D(A), |u|=1} (Au, u).$$

We also introduce

$$V = D(A^{1/2})$$

endowed with the norm given by

$$\forall u \in V, \quad \|u\|^2 = |A^{1/2}u|^2.$$

We recall that

$$\forall u \in D(A), \quad |A^{1/2}u|^2 = (Au, u).$$

We consider the second order evolution equation

$$u'' + cu' + Au = f(t) \tag{3.1}$$

where $f \in L^\infty(\mathbb{R}, H)$ and $c > 0$.
A. Haraux [13] established that the bounded solution u of (3.1) satisfies the estimates

$$\|u(t)\| \ \leq \sqrt{\frac{4}{c^2} + \frac{1}{\lambda_1}} \ \ \|f\|_{L^\infty(R,H)} \leq \frac{2\sqrt{2}}{c}\|f\|_{L^\infty(R,H)}, \quad \forall t \in \mathbb{R} \tag{3.2}$$

$$|u'(t)| \leq \frac{4}{c}\|f\|_{L^\infty(R,H)}, \quad \forall t \in \mathbb{R} \tag{3.3}$$

in the case $c \leq 2\sqrt{\lambda_1}$, and satisfies the estimate

$$\|u(t)\| \ \leq \sqrt{\frac{8}{c^2} + \frac{2}{\lambda_1}}\,\|f\|_{L^\infty(R,H)} \leq \frac{2}{\sqrt{\lambda_1}}\,\|f\|_{L^\infty(R,H)}, \quad \forall t \in \mathbb{R} \tag{3.4}$$

in the case $c > 2\sqrt{\lambda_1}$.
Assume moreover that A is bounded and that

$$c \geq 2\|A^{\frac{1}{2}}\|. \tag{3.5}$$

Then, the estimate of u is

$$\forall t \in \mathbb{R}, \ \|u(t)\| \leq \sqrt{\frac{4}{c^2} + \frac{1}{\lambda_1}}\|f\|_{L^\infty(\mathbb{R},H)} \leq \sqrt{\frac{2}{\lambda_1}}\|f\|_{L^\infty(\mathbb{R},H)}. \tag{3.6}$$

In this chapter, we obtain the same estimate as (3.6) for the large damping case $c > 2\sqrt{\lambda_1}$, after removing the assumption (3.5). We thus impove the estimate of A. Haraux [13] by a factor $\sqrt{2}$.

2 Ultimate bound for c large

Theorem 3.1. *Assuming $c \geq 2\sqrt{\lambda_1}$ or equivalently*

$$\lambda_1 < \frac{c^2}{4},$$

the bounded solution u of (3.1) *satisfies the estimate* (3.6) *and*

$$\forall t \in \mathbb{R}, \quad |u'(t)| \leq \frac{\sqrt{2}c}{\sqrt{\lambda_1(2\lambda_1 + c\sqrt{c^2 - 4\lambda_1})}}\|f\|_{L^\infty(\mathbb{R},H)}. \tag{3.7}$$

Proof. We introduce

$$\Phi(t) := |u'(t)|^2 + \|u(t)\|^2 + \alpha(u(t), u'(t)),$$

where $\alpha = c - \sqrt{c^2 - 4\lambda_1}$. We have

$$\begin{aligned}\Phi'(t) &=< u'' + Au, 2u' > +\alpha|u'|^2 + \alpha < u'', u >\\ &= (\alpha - 2c)|u|'^2 + \alpha < f - Au - cu', u > +2(f, u')\\ &= -\frac{\alpha}{2}\Phi + (-c\alpha + \frac{\alpha^2}{2})(u, u') + (\frac{3\alpha}{2} - 2c)|u'|^2 - \frac{\alpha}{2}\|u\|^2 + (f, 2u' + \alpha u).\end{aligned}$$

On the other hand

$$|2u' + \alpha u|^2 = 4|u'|^2 + 4\alpha(u, u') + \alpha^2|u|^2 \le 4|u|^2 + 4\alpha(u, u') + \frac{\alpha^2}{\lambda_1}\|u\|^2.$$

Hence, using

$$(f, 2u' + \alpha u) \le \frac{\alpha}{2\lambda_1}|f|^2 + \frac{\lambda_1}{2\alpha}[4|u'|^2 + \alpha^2|u|^2 + 4\alpha(u, u')],$$

we deduce the inequality

$$\Phi'(t) \le -\frac{\alpha}{2}\Phi(t) + (-c\alpha + \frac{\alpha^2}{2} + 2\lambda_1)(u, u') + (\frac{3\alpha}{2} - 2c + \frac{2\lambda_1}{\alpha})|u'|^2 + \frac{\alpha}{2\lambda_1}|f|^2,$$

hence

$$\Phi'(t) \le -\frac{\alpha}{2}\Phi(t) + \frac{\alpha}{2\lambda_1}|f|^2.$$

In particular, since Φ is bounded we find

$$\forall t \in \mathbb{R}, \quad \Phi(t) \le \frac{1}{\lambda_1}\|f\|^2,$$

which means

$$\forall t \in \mathbb{R}, \quad |u'(t)|^2 + \|u(t)\|^2 + \alpha(u(t), u'(t)) \le \frac{1}{\lambda_1}\|f\|_\infty^2. \tag{3.8}$$

In particular

$$\forall t \in \mathbb{R}, \quad \lambda_1|u(t)|^2 + \alpha(u(t), u'(t)) \le \frac{1}{\lambda_1}\|f\|_\infty^2$$

and this means

$$\frac{\alpha}{2}(|u(t)|^2)' + \lambda_1|u(t)|^2 \le \frac{1}{\lambda_1}\|f\|_\infty^2.$$

Along with boundedness of $u(t)$ in H on $\mathbb{R}$ this implies

$$\forall t \in \mathbb{R}, \quad |u(t)|^2 \leq \frac{1}{\lambda_1^2}\|f\|_\infty^2. \tag{3.9}$$

Finally from (3.8) and since $\alpha \leq \frac{4\lambda_1}{c}$ we deduce that

$$\forall t \in \mathbb{R}, \quad \|u(t)\|^2 \leq \frac{1}{\lambda_1}\|f\|_\infty^2 - |u'(t)|^2 - \alpha(u(t), u'(t)) \leq \frac{1}{\lambda_1}\|f\|_\infty^2 + \frac{4{\lambda_1}^2}{c^2}|u(t)|^2$$

$$\leq (\frac{4}{c^2} + \frac{1}{\lambda_1})\|f\|_\infty^2,$$

hence (3.6) is established. To check (3.7) we start from (3.8)-(3.9) which provide

$$\forall t \in \mathbb{R}, \quad |u'(t)|^2 + 2\lambda_1|u(t)|^2 + \alpha(u(t), u'(t)) \leq \frac{2}{\lambda_1}\|f\|_\infty^2.$$

In particular

$$\forall t \in \mathbb{R}, \quad |u'(t)|^2 + \frac{c^2}{2}|u(t)|^2 + \alpha(u(t), u'(t)) \leq \frac{2}{\lambda_1}\|f\|_\infty^2.$$

Hence

$$\forall t \in \mathbb{R}, \quad |u'(t)|^2 \leq \frac{2}{\lambda_1}\|f\|_\infty^2 - \frac{c^2}{2}|u(t)|^2 - \alpha(u(t), u'(t)) \leq \frac{2}{\lambda_1}\|f\|_\infty^2 + \frac{\alpha^2}{2c^2}|u'(t)|^2$$

from which (3.7) follows at once. $\diamond$

Chapter 4

Sharp estimates of bounded solutions to some semilinear second order dissipative equations

1 Introduction

We denote by (u, v) the inner product of two vectors u, v in H (Hilbert space) and by $|u|$ the H-norm of u. We consider a second Hilbert space $V \subset H$ with continuous and dense imbedding and we denote by $\|u\|$ the V-norm of u. The duality pairing between $\varphi \in V'$ and $u \in V$ is denoted by $\langle f, u\rangle$. We identify H with its dual which implies $H \subset V'$ and the identity

$$\forall u \in H,\ \forall v \in V, \quad \langle u, v\rangle = (u, v).$$

Let b, c be two positive constants and $F \in C^1(V)$ be **convex, nonnegative**. The equation

$$u'' + cu' + bu + \nabla F(u) = f(t) \tag{4.1}$$

is a natural vector generalization of the scalar ODE

$$u'' + cu' + g(u) = f(t) \tag{4.2}$$

considered after, [4], in [16] (cf. also [9] for a pure differential inequality treatment) under the hypothesis

$$g \in C^1\,,\ g' \geq b > 0. \tag{4.3}$$

In addition when F is a nonnegative quadratic form on V, equation (4.1) becomes the equation

$$u'' + cu' + Au = f(t) \tag{4.4}$$

where $A = bI + \nabla F$ is a linear self-adjoint operator and $A \geq bI$. The results of this chapter extend some results from both [16] and [13] on the ultimate bound of solutions to (4.2) and (4.4) respectively. In addition the result of [13] is improved for c large. This comes from a different proof based on a new energy functional which allows the extension to the case of a nonlinear strongly monotone conservative term.

2 Main results.

The following general result will be established in Sections 3 and 4.

Theorem 4.1. *Let b, c be two positive constants and $F \in C^1(V)$ be* **nonnegative and convex**. *Then for any solution $u \in C^1(\mathbb{R}^+, V) \cap W^{2,\infty}_{loc}(\mathbb{R}^+, V')$ of (4.1), u is bounded with values in H and*

$$\overline{\lim_{t\to+\infty}} |u(t)| \leq \max\{\frac{1}{b}, \frac{2}{c\sqrt{b}}\} \overline{\lim_{t\to+\infty}} |f(t)|. \tag{4.5}$$

Moreover, introducing for each $u \in V$

$$G(u) = \frac{b}{2}|u|^2 + F(u),$$

$G(u(t))$ is bounded on $\mathbb{R}^+$ with the estimate

$$2 \overline{\lim_{t\to+\infty}} G(u(t)) \leq (\frac{4}{c^2} + \frac{1}{b}) \overline{\lim_{t\to+\infty}} |f(t)|^2. \tag{4.6}$$

In addition for $c \leq 2\sqrt{b}$,

$$\overline{\lim_{t\to+\infty}} |u'(t)| \leq (\frac{2}{c} + \frac{1}{c\sqrt{b}}) \overline{\lim_{t\to+\infty}} |f(t)| \leq \frac{4}{c} \overline{\lim_{t\to+\infty}} |f(t)| \tag{4.7}$$

and for $c > 2\sqrt{b}$,

$$\overline{\lim_{t\to+\infty}} |u'(t)| \leq \frac{c\sqrt{2}}{\sqrt{b(c^2-2b)}} \overline{\lim_{t\to+\infty}} |f(t)| \leq \frac{2}{\sqrt{b}} \overline{\lim_{t\to+\infty}} |f(t)|. \tag{4.8}$$

Remark 4.1. *In the limiting case $c = 2\sqrt{b}$, the four constants in (4.7) and (4.8) are equal:*

$$\frac{2}{c} + \frac{1}{c\sqrt{b}} = \frac{4}{c} = \frac{c\sqrt{2}}{\sqrt{b(c^2-2b)}} = \frac{2}{\sqrt{b}}.$$

On the other hand when $\frac{c}{\sqrt{b}} \longrightarrow 0$ the left constant is equivalent to $\frac{2}{c}$ and when $\frac{c}{\sqrt{b}} \longrightarrow +\infty$ the constant $\frac{c\sqrt{2}}{\sqrt{b(c^2-2b)}}$ is equivalent to $\frac{\sqrt{2}}{\sqrt{b}}$. However (4.8) is weak compared to *the estimate given in [16] who found $\frac{4}{c}$ in all cases, for the scalar equation (4.2). We do not know whether the same result is true for the general equation (4.1).*

In the applications it is sometimes useful to consider the slightly different situation of solutions defined and bounded on the whole real line. This is the object of our second result.

Theorem 4.2. *Let b, c and F, G be as in the statement of Theorem 4.1. Then for any solution $u \in C_b(\mathbb{R}, V) \cap C_b^1(\mathbb{R}, H) \cap W_{loc}^{2,\infty}(\mathbb{R}, V')$ of (1.1), the following estimates are valid*

$$\forall t \in \mathbb{R}, \quad |u(t)| \le \max\{\frac{1}{b}, \frac{2}{c\sqrt{b}}\} \|f\|_{L^\infty(\mathbb{R}, H)}. \tag{4.9}$$

$$\forall t \in \mathbb{R}, \quad 2G(u(t)) \le (\frac{4}{c^2} + \frac{1}{b}) \|f\|_{L^\infty(\mathbb{R}, H)}. \tag{4.10}$$

In addition for $c \le 2\sqrt{b}$,

$$\forall t \in \mathbb{R}, \quad |u'(t)| \le (\frac{2}{c} + \frac{1}{c\sqrt{b}}) \|f\|_{L^\infty(\mathbb{R}, H)} \le \frac{4}{c} \|f\|_{L^\infty(\mathbb{R}, H)}, \tag{4.11}$$

and for $c > 2\sqrt{b}$,

$$\forall t \in \mathbb{R}, \quad |u'(t)| \le \frac{c\sqrt{2}}{\sqrt{b(c^2 - 2b)}} \|f\|_{L^\infty(\mathbb{R}, H)} \le \frac{2}{\sqrt{b}} \|f\|_{L^\infty(\mathbb{R}, H)}. \tag{4.12}$$

3 Proof in the case of a small damping.

This section is devoted to the proof of Theorems 4.1 and 4.2 under the hypothesis

$$c \le 2\sqrt{b}. \tag{4.13}$$

In this case we can use the following energy functional :

$$\Phi(t) = |u'(t)|^2 + 2G(u(t)) + c(u(t), u'(t)). \tag{4.14}$$

Here we have, setting

$$g(u) := bu + \nabla F(u) = \nabla G(u),$$

$$\Phi' = \langle u'' + g(u), 2u' \rangle + c|u'|^2 + c\langle u'', u \rangle = -c|u'|^2 + c\langle f - g(u) - cu', u \rangle + 2(f, u')$$

$$\Phi' = -c(u'^2 + \langle g(u), u \rangle + c(u, u')) + (f, 2u' + cu). \tag{4.15}$$

By convexity of F we have on the other hand

$$\forall u \in V, \quad \langle g(u), u \rangle = b|u|^2 + \langle \nabla F(u), u \rangle \ge b|u|^2 + F(u) = \frac{b}{2}|u|^2 + G(u).$$

Hence

$$(u'^2 + \langle g(u), u\rangle + +c(u,u')) \geq \frac{1}{2}(u'^2 + 2G(u) + +c(u,u')) + \frac{1}{2}(u'^2 + b|u|^2 + c(u,u')).$$

Therefore (4.15) implies

$$\Phi'(t) \leq -\frac{c}{2}\Phi(t) - \frac{c}{2}(u'^2 + b|u|^2 + c(u,u')) + f(2u' + cu). \tag{4.16}$$

On the other hand since $2G(u) \geq b|u|^2$ we have by (4.13)

$$|2u' + cu|^2 = 4|u'|^2 + 4c(u,u') + c^2|u|^2 \leq 4|u'|^2 + 4c(u,u') + 4b|u|^2 \leq 4\Phi.$$

Hence, using

$$(f, 2u' + cu) \leq \frac{2}{c}|f|^2 + \frac{c}{8}|2u' + cu|^2 \leq \frac{2}{c}|f|^2 + \frac{c}{2}\Phi,$$

we deduce from (4.16) the inequality

$$\Phi' \leq -\frac{c}{2}\Phi + \frac{2}{c}|f|^2. \tag{4.17}$$

In particular we find that Φ is bounded with

$$\overline{\lim_{t\to+\infty}}\,\Phi(t) \leq \frac{4}{c^2}\overline{\lim_{t\to+\infty}}\,|f(t)|^2.$$

Fix any number

$$A > \frac{4}{c^2}\overline{\lim_{t\to+\infty}}\,|f(t)|^2.$$

Then for t large enough we have

$$|u'(t)|^2 + 2G(u(t)) + c(u(t), u'(t)) \leq A. \tag{4.18}$$

In particular for t large enough

$$b|u(t)|^2 + c(u(t), u'(t)) \leq A,$$

and this means

$$\frac{c}{2}(|u(t)|^2)' + b|u(t)|^2 \leq A.$$

In particular

$$b\,\overline{\lim_{t\to+\infty}}\,|u(t)|^2 \leq A,$$

and by minimizing A we deduce

$$b\,\overline{\lim_{t\to+\infty}}\,|u(t)|^2 \leq \frac{4}{c^2}\overline{\lim_{t\to+\infty}}\,|f(t)|^2. \tag{4.19}$$

Finally from (4.17) we deduce for any A as above and all t large enough

$$2G(u(t)) \leq A - |u'(t)|^2 - c(u(t), u'(t)) \leq A + \frac{c^2}{4}|u(t)|^2$$

and then (4.6) follows from (4.19). To check (4.7) we start from (4.18) and (4.19) which give

$$|u'(t)|^2 + b|u(t)|^2 + c(u(t), u'(t)) \leq A,$$

valid for t large enough. In particular for t large:

$$|u'(t) + \frac{c}{2}u(t)|^2 = |u'(t)|^2 + \frac{c^2}{4}|u(t)|^2 + c(u(t), u'(t)) \leq A.$$

Hence

$$|u'(t)| \leq |u'(t) + \frac{c}{2}u(t)| + \frac{c}{2}|u(t)| \leq A^{\frac{1}{2}} + \frac{c}{2\sqrt{b}}A^{\frac{1}{2}}$$

from which (4.7) follows at once by letting

$$A \longrightarrow \frac{4}{c^2} \overline{\lim_{t\to+\infty}} |f(t)|^2.$$

The proof of Theorem 4.2 follows the same steps but at each stage the inequalities are valid for all $t \in \mathbb{R}$ and the upper limits are replaced by uniform bounds.

4 Proof in the case of large damping.

This section is devoted to the proof of Theorems 4.1 and 4.2 under the hypothesis

$$c \geq 2\sqrt{b}. \tag{4.20}$$

In this this case we can use the same form of energy functional as in chapter 2 which is:

$$\Phi(t) := |u'(t)|^2 + 2G(u(t)) + \alpha(u(t), u'(t)). \tag{4.21}$$

We have

$$\Phi'(t) = \langle u'' + g(u), 2u'\rangle + \alpha|u'|^2 + \alpha\langle u'', u\rangle = (\alpha - 2c)|u'|^2 + \alpha\langle f - g(u) - cu', u\rangle + 2(f, u')$$

$$\Phi'(t) = (\alpha - 2c)|u'|^2 + (f, 2u' + \alpha u) - \alpha\langle g(u), u\rangle - \alpha c(u, u').$$

Since $\langle g(u), u\rangle \geq \frac{b}{2}|u|^2 + G(u)$ we obtain

$$\Phi'(t) \leq (\alpha - 2c)|u'|^2 - \alpha G(u) - \alpha\frac{b}{2}|u|^2 - c\alpha(u, u') + (f, 2u' + \alpha u).$$

Hence

$$\Phi'(t) \leq -\frac{\alpha}{2}\Phi(t) + (\frac{3\alpha}{2} - 2c)|u'|^2 - \alpha\frac{b}{2}|u|^2 + (-c\alpha + \frac{\alpha^2}{2})(u, u') + (f, 2u' + \alpha u).$$

On the other hand we have

$$(f, 2u' + \alpha u) \leq \frac{\alpha}{2b}|f|^2 + \frac{b}{2\alpha}(4|u'|^2 + \alpha^2|u|^2 + 4\alpha(u, u'))$$

therefore

$$\Phi'(t) \leq -\frac{\alpha}{2}\Phi(t) + (\frac{3\alpha}{2} - 2c + \frac{2b}{\alpha})|u'|^2 + (-c\alpha + \frac{\alpha^2}{2} + 2b)(u, u') + \frac{\alpha}{2b}|f|^2.$$

Hence

$$\Phi'(t) \leq -\frac{\alpha}{2}\Phi(t) + \frac{\alpha}{2b}|f|^2.$$

In particular, we find that Φ is bounded with

$$\overline{\lim_{t\to\infty}}\, \Phi(t) \leq \frac{1}{b}\overline{\lim_{t\to\infty}}\, |f|^2.$$

Fix any number A

$$A > \frac{1}{b}\overline{\lim_{t\to\infty}}\, |f|^2.$$

Then for t large enough we have

$$|u'(t)|^2 + 2G(u(t)) + \alpha(u(t), u'(t)) \leq A. \tag{4.22}$$

In particular

$$b|u(t)|^2 + \alpha(u(t), u'(t)) \leq A,$$

and this means

$$\frac{\alpha}{2}(|u(t)|^2)' + b|u(t)|^2 \leq A.$$

In particular

$$b\,\overline{\lim_{t\to\infty}}\, |u(t)|^2 \leq A,$$

and by minimizing A we deduce

$$b\,\overline{\lim_{t\to\infty}}\, |u(t)|^2 \leq \frac{1}{b}\overline{\lim_{t\to\infty}}\, |f|^2. \tag{4.23}$$

Finally from (4.22) and since $\alpha \leq \dfrac{4b}{c}$ we deduce for any A as above and all t large enough

$$2G(u(t)) \leq A - |u'(t)|^2 - \alpha(u(t), u'(t)) \leq A + \frac{4b^2}{c^2}|u(t)|^2,$$

and then (4.6) follows from (4.23). To check (4.8) we start from (4.22) and (4.23) wich give

$$|u'(t)|^2 + 2b|u(t)|^2 + \alpha(u(t), u'(t)) \leq 2A,$$

valid for t large enough. Hence

$$|u'(t)|^2 \leq 2A - 2b|u(t)|^2 - \alpha(u(t), u'(t)) \leq 2A + \frac{\alpha^2}{8b}|u'(t)|^2.$$

On the other hand we have

$$\alpha = c - \sqrt{c^2 - 4b} = \frac{4b}{c + \sqrt{c^2 - 4b}}.$$

Therefore

$$\frac{\alpha^2}{8b} = \frac{2b}{(c + \sqrt{c^2 - 4b})^2} \leq \frac{2b}{c^2},$$

so that we obtain

$$|u'(t)|^2 \leq 2A + \frac{2b}{c^2}|u'(t)|^2$$

from wich (4.8) follows at once by letting

$$A \longrightarrow \frac{1}{b} \overline{\lim_{t \to +\infty}} |f(t)|^2.$$

The proof of Theorem 4.2 follows the same steps but at each stage the inequalities are valid for all $t \in \mathbb{R}$ and the upper limits are replaced by uniform bounds.

5 Applications.

As mentioned in the introduction, Theorems 4.1 and 4.2 now enable us to improve several boundedness results which appeared previously in the Literature.

5.1 Application to Duffing's equation.

When we apply Theorem 4.1 to the Duffing equation, we obtain immediately the estimations cited in the second chapter for any solution in $C^1(\mathbb{R}^+) \cap W^{2,\infty}_{loc}(\mathbb{R}^+)$ of (4.2).

5.2 The case of linear evolution equations.

Let consider the second order evolution equation (4.1) which is given in the third chapter. It is well-known that this equation have a unique bounded solution $u \in C_b(\mathbb{R}, V) \cap C^1_b(\mathbb{R}, H)$, which attracts exponentially all solutions (and in particular all strong solutions)as t goes to infinity. As a consequence of Theorem 4.7 associated with a density argument for smooth forcing terms f we obtain the estimations of chapter 3.

5.3 Attractors of semilinear hyperbolic problems.

Let Ω be a bounded open domain in $\mathbb{R}^N$ and $b \geq 0, c > 0$. We consider the problem

$$u'' - \Delta u + g(u) + cu' = a \sin u, \tag{4.24}$$

with one of the boundary conditions

$$u = 0 \quad \text{on } \partial\Omega \tag{4.25}$$

or

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{4.26}$$

$g \in C^1$ is such that for some nonnegative constants b, C, γ we have

$$\forall s \in \mathbb{R}, \quad b \leq g'(s) \leq C(1 + |s|)^\gamma. \tag{4.27}$$

It is well known that under the growth condition (4.27) with

$$(N-2)\gamma < 2, \tag{4.28}$$

problems (4.24)-(4.25) and (4.24)-(4.26) have unique solutions for given initial data in the energy space and these solutions can be approximated, cf e.g. [13], by solutions which satisfy the regularity conditions $u \in C^1(\mathbb{R}^+, V) \cap W^{2,\infty}_{loc}(\mathbb{R}^+, V')$ where $V = H^1_0(\Omega)$ in the first case and $V = H^1(\Omega)$ in the second one. In addition the dynamical system generated by (4.24) is well known to have a compact attractor $\mathcal{A}$ under the condition $b > 0$ in the second case. The result of Theorem 4.1 now gives the following upper bound of the size of the u-projection of $\mathcal{A}$.

Corollary 4.1. *In the case of problem (4.24)-(4.25) we have*

$$\forall (u,v) \in \mathcal{A}, \quad \left\{ \int_\Omega (\|\nabla u\|^2 + bu^2) dx \right\}^{1/2} \leq a|\Omega|^{1/2} \sqrt{\frac{4}{c^2} + \frac{1}{\lambda_1(\Omega) + b}}, \tag{4.29}$$

and for problem (4.24)-(4.26) we have

$$\forall (u,v) \in \mathcal{A}, \quad \left\{ \int_\Omega (\|\nabla u\|^2 + bu^2) dx \right\}^{1/2} \leq a|\Omega|^{1/2} \sqrt{\frac{4}{c^2} + \frac{1}{b}} \tag{4.30}$$

These estimates generalize a result from [13] and are, surprisingly enough, close to optimality even when g is linear, as was shown in [13]. Theorem 4.1 also provides the corresponding estimates on $v = u'$ but they are less interesting and probably not quite optimal. ◇

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Curriculum Vitae

Surname:	Fitouri
First name:	Cyrine
Birth date:	June 10, 1979
Nationality:	Tunisian

Education:

Sep.1993 - Jun.1997:	Lyceum Père Blanc, Tunis.
Sep.1998 - Jul.2002:	Master's Degree in Mathematics Faculty of Sciences of Tunis, Tunisia.
Sep.2002 - Oct.2003:	Postgraduate Diploma in Mathematics University Denis Diderot (Paris 7), France.
Nov.2003 - Oct.2006:	Ph.D. student under supervision of Prof. Dr. Alain Haraux, Laboratoire J.L. Lions, University Pierre et Marie Curie (Paris 6), France.
Nov.2006 - Feb.2008:	Teaching assistent and Ph.D. student under supervision of Prof. Dr. Michel Chipot, at the Institute of Mathematics, University of Zürich, Switzerland.